EXTENDED RESULTS IN PENDANT DOMINATION POLYNOMIAL OF A GRAPH

Pavithra G B, Swaroopa Rani N C
Assistant Professor
Mysore University

Abstract

Let $G = (V, E)$ be a graph. A subset of S of vertices in G is called a pendant dominating set if the sub graph induced by S contains a pendant vertex. The pendant domination polynomial of G is defined as $D_{pe}(G,x) = \sum_{i=1}^{n} d_{pe}(G,i)x^i$, where $d_{pe}(G,i)$ is the number of pendant dominating sets of size i. In this paper, we extend the study of pendant domination polynomial of a graph and recurrence relations are obtained for polynomials of path and cycle graphs

Keywords : Domination, Pendant Domination, Domination Polynomial

1. Introduction

Let $G = (V, E)$ be any graph of order n and size m. A vertex of degree zero is called an isolated vertex and a vertex of a degree one is called a pendant vertex. An edge incident to pendant vertex is called a pendant edge. For a detailed treatment of the domination polynomial of a graph, the reader is referred to [1]. We have introduced and studied the concept of pendant domination polynomial of G, and obtained pendant domination polynomial for some standard graphs in [4]. The join of G and H denoted by $G \vee H$ is the graph such that $G \vee H = V(G) \cup V(H)$ and $E(G \vee H) = E(G) \cup E(H) \cup \{uv: u \in V(G), v \in V(H)\setminus\}$. Let G_1 and G_2 be sub graphs of G. We say that G_1 and G_2 are disjoint if they have no vertex in common and edge-disjoint if they have no edge in common. The union $G_1 \cup G_2$ of G_1 and G_2 is the subgraph with vertex set $V(G_1) \cup V(G_2)$ and edge set $E(G_1) \cup E(G_2)$; if G_1 and G_2 are disjoint, we sometimes denote their union by $G_1 \vee G_2$. A graph G is called a bi-star if it can be constructed from K_2 by touching m edges in one vertex and n in the other vertex, denoted by $B(m,n)$. Graph polynomial is good representation for the graph. There are many polynomials which play an important role in Combinatorics. One of the recent graph polynomial is domination polynomial which was introduced in [1]. The dominating polynomial motivated us to introduce and study the pendant domination polynomial in graph.

Dominating set: A subset S of V (G) is a dominating set G if each vertex $u \in V - S$ is adjacent to a vertex in S. The least cardinality of a domination set is called the domination number of G, donated by $\gamma(G)$

Isolated Dominating Set: A subset S of V (G) is called an isolated dominating set of G if < S > contains at least one isolated vertex. The minimum cardinality of an isolated dominating set in G is called the isolated domination number of G, denoted by $\gamma_0(G)$.

Pendant Domination Number: A dominating set S in G is called a pendant dominating set if < S > contains at least one pendant vertex. The minimum cardinality of a pendant dominating set is called the pendant domination number denoted by $\gamma_{pe}(G)$. The pendant domination polynomial is defined for all non-trivial connected graphs of order at least two. Hence, throughout the paper we assume that by a graph G, we mean a connected graph of order at least two.

II. Pendant Domination Polynomial of a graph:

Let $G = (V, E)$ be any graph of order $n \geq 2$. The pendant domination polynomial of G is denoted by $D_{pe}(G,x)$ and is defined as $D_{pe}(G,x) = \sum_{i=1}^{n} d_{pe}(G,i)x^i$, where $d_{pe}(G,i)$ is the number of pendant dominating sets of size i.

The pendant domination polynomial is defined for all non-trivial connected graphs of order at least two. Hence, throughout the paper we assume that by a graph G, we mean a connected graph of order at least two.

Observations 1:1: Let G be any connected graph of order $n \geq 2$, Then

1. $d_{pe}(G,n) = \begin{cases} 1, & if\ \delta(G) = 1 \\ 0, & otherwise \end{cases}$
2. $d_{pe}(G,i) = 0$ if only if $i < \gamma_{pe}(G)\ or\ i > n$
3. $D_{pe}(G,x)$ is a monotonically increasing function.
4. $D_{pe}(G,x)$ has no constant.
5. $zero$ is a root of $D_{pe}(G,x)\ of$ multiplicity $\gamma_{pe}(G)$.

Theorem 1.1: if a graph G consists of m components $G_1, G_2, \ldots\ldots G_m, then\ D(G,x) = D(G_1,x)D(G_2,x)\ldots\ldots D(G_m,x)$

Proof: Let G be a disconnected graph having components. Then, the pendant domination polynomial of each connected component will be a factor of pendant domination polynomial of G. Therefore, pendant domination polynomial of G is obtained by considering the product of pendant domination polynomial of each component.

Proposition 1.1[4]: Let G be a connected graph of order $n \geq 2$. Then $D_{pe}(G,x) = \binom{n}{2}x^2$ if only if G is a complete graph of order n.

Theorem 1.2[4]: Let G_1, G_2 be connected graphs of order $n, m \geq 2$ respectively. Then $D_{pe}(G_1 \vee G_2, x) = nmx^2 + D_{pe}(G_1,x) + D_{pe}(G_2,x)$.

Theorem 1.3: Let G_1, G_2 be connected graphs of order $n, m \geq 2$ respectively. Then $D_{pe}(G_1 \cup G_2, x) = D_{pe}(G_1,x) + D_{pe}(G_2,x)$.

Proposition 1.2. The following properties hold for coefficients of $D_{pe}(C_n,x)$.

1. $d_{pe}(C_n,n) = 1, if\ n \geq 2.$
2. $d_{pe}(C_n,n-1) = n, if\ n \geq 3.$
3. $d_{pe}(C_{3n},n) = 0.$
4. $d_{pe}(C_{3n},n+1) = n(n+1)$
5. $d_{pe}(C_{3n+1},n+1) = n$
6. $d_{pe}(C_{4n},2n) = 1$

Proposition 1.3. The following properties hold for coefficients of $D_{pe}(P_n,x)$.

1. $d_{pe}(P_n,n) = 1, if\ n \geq 2.$
2. $d_{pe}(P_n,n-1) = n, if\ n \geq 3.$
3. $d_{pe}(P_{3n},n) = 0.$
4. $d_{pe}(P_{3n},n+1) = n(n+1)$
5. $d_{pe}(P_{3n+1},n+1) = n$
6. $d_{pe}(P_{4n},2n) = 1$

Theorem 1.4[4]: Let $G \cong K_{1,n}$ be a star of order n. Then $D_{pe}(G,x) = x((1+x)^{n-1} - 1))$. Converse of the above result is also true. We state above theorem as follows:

Theorem 1.5: Let G be any connected graph n. Then $D_{pe}(G,x) = x((1+x)^{n-1} - 1))$ if and only if G is a star on n vertices.

Proof: Let G be a connected graph order n with the pendant domination polynomial $D_{pe}(G,x) = x((1+x)^{n-1} - 1))$. It is clear from the polynomial that, there are exactly n-1 pendant dominating sets of cardinality 2. Since, pendant dominating set of size two leads to an edge in G, it follows that G contains at least n-1 edges. For any integer $k, 2 \leq k \leq n$, the number of pendant dominating set of cardinality k is given by $\binom{n-1}{k-1}$. Thus a pendant dominating set of size k is obtained by choosing k-1 vertices out of n-1 vertices. Therefore, G is isomorphic to the sum of K_1 and a graph of order n-1 say H. That is, $G \cong K_1 \vee H$. Clearly H must be totally disconnected. On contrary, suppose H is connected. It is possible to choose a pendant dominating S of H of cardinality say k. Then S is also a pendant dominating set of G of cardinality k. This implies that $d_{pe}(G,k)$ is at least $\binom{n}{k} + 1$, which is a contradiction. This contradiction show that $H \cong \bar{k}_{n-1}$ and hence $G \cong K_1 \vee \bar{k}_{n-1}$, a star on n vertices. Converse follows from Theorem 1.4.

Theorem 1.6: Let G be any connected graph $m + n + 2$ vertices. Then $D_{pe}(G,x) = x^2(1+x)^{m+n}$ if and only if G is a bi-star on n vertices

Proof: Let G be a connected graph of order $m + n + 2$ with the pendant domination polynomial $D_{pe}(G,x) = x^2(1+x)^{m+n}$. It is clear from the polynomial that, there is only one pendant dominating set of cardinality 2. Therefore, \$G\$ contains only one edge $e = uv$ of degree at least $m + n$. That is, every vertex in G is adjacent to u or v. For if all the vertices in G are adjacent to u only, then for any vertex w different from v, the edge $e' = uw$ will be a pendant dominating set in G, which is not possible. Thus, both u ad v are of degree at least two. Suppose S_1, S_2 be any two subsets of $V(G)$of cardinality m, n respectively, containing neither u nor v.

From the polynomial, it is clear that, there are $\binom{m+n}{k-2}$ways to select pendant dominating sets of cardinality k by choosing vertices from S_1 and S_2 together with u, v. Now, we show that no vertex in S_1 is adjacent to a vertex in S_2. On contrary, if there is an edge between $S_1$and S_2, then we any pendant dominating set of $< V(G) - \{u,v\} >$ of cardinality k will be a pendant dominating set of G. Therefore, there at least $\binom{m+n}{k-2} + 1$ possibilities for a pendant dominating sets of cardinality k, a contradiction. This contradiction proves that, there are no edges between S_1 and S_2. Further, it is easy to observe that the sub graphs induced S_1 and S_2 are totally disconnected. Therefore, G obtained by attaching m, n number of vertices to u, v respectively. That is, $G \cong B(m,n)$.

Theorem 1.7: Let G be a graph of connected graph of order $n \geq 3$ with r isolated vertices. Then

1. $r = n - d_{pe}(G, n-1)$
2. $d(G,2) = |\{uv \in E(G) | deg(e) \geq 2(n-2) and\ N(v) \cap N(u) = \emptyset\}|$

Proof:

1. Let G be a connected graph and let A be a collection of isolated vertices in G. Therefore by our assumption, $|A| = r$. Now for any vertex $v \in V(G) - A$, the set $V(G) - \{v\}$ is a pendant dominating set of G. Therefore, $d_{pe}(G, n-1) = |V(G-A)| = n - r$, and $r = n - d_{pe}(G, n-1)$.
2. A pendant dominating set of cardinality is an edge $e = uv$ in G. Therefore $d_{pe}(G,2)$$ counts number of edges in G dominating G. That is, number of edges $e = uv$ with $\deg(e) = \deg(u) + \deg(v)$ is at least $2(n-2)$ such that $N(v) \cap N(u) = \emptyset$.

A finite sequence of real numbers $\{d_0, d_1, \ldots\ldots d_m\}$ is said to be unimodal if there exist an index $0 \leq j \leq m$ such that $d_0 \leq d_1 \leq \cdots .. \leq d_j$ and $d_j \geq d_{j+1} \geq \cdots .. \geq d_m$$. A polynomial is said to be unimodal if its sequence of coefficients is unimodal.

Conjecture: The domination polynomial of any graph is unimodal.

When counting the number of dominating sets with cardinality close to n, it may simplify things to count the number of subsets which are not dominating. A subset $S \subseteq V(G)$ is not dominating if there exists a vertex v in G such that none of its neighbors, nor itself, is in S. That is, $N[v] \cap S = \phi$. The next proposition will help us identify which subsets are not dominating by looking at the subset's complement (with respect to V). If there exists a vertex v and subset $S \subseteq V$ such that $N[v] \subseteq S$ then we say S encompasses v or v is encompassed by S. We shall state the following proposition without proof.

Proposition 1.4: For a graph G and $S \subseteq V(G)$. S is not dominating if and only if there exists a vertex $v \in S$ which is encompassed by S.

III. Conclusion

In recent days, the study of domination and its related parameters has a major significance in graph theory and polynomial representation of graphs leads to study of graphs through graphs. In this article, we are interested in studying behavior of roots of pendant domination polynomial. We also study the unimodality of pendant domination polynomial and try to obtain expressions for $d_{pe}(G, n-2)$ and $d_{pe}(G, n-3)$.

IV. References

1. Saied Alikhani and Yee-hock Peng, Introduction to domination polynomial of a graph, Ars Combin. 114(2014) 257-266
2. T.W.Haynes, S.T.Hedetniemi, P.J.Slater, Fundamentals of Domination in Graphs, Marcel Dekkar, Newyork 1998
3. T.W.Haynes, H.T.Hedetniemi and P.J.Slater, Domination in Graphs: Advanced Topics, (Marcel Dekkar, Newyork, 1998)
4. Nayaka S.R.Puttaswamy and Purushothama S., Pendant Domination Polynomial of a Graph, International Journal of Pure ad Applied Mathematics, Volume 117, No. 11, (2017) 193-199.