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Construction of solvable potentials from Generalized Hulthen Potential using Green's Function technique

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Abstract: We have applied extended transformation method to an exactly solvable non power multi-term potential to generate a set of exactly solved quantum systems in any chosen dimensional Euclidean space. The bound state and scattering state S-wave solution of Schrodinger Green's function equation of the constructed quantum systems are reported. Taking the first term of the parent potential as working potential the generated potential behaves like a potential qualitatively similar to the Rosen-Morse potential. The generated potential, taking the second term as working potential belongs to the family of the Poschl-Teller potential and the third working potential reverse back to the original potential. Existence of normalizable eigenfunctions for these systems found to be dependent on the local and asymptotic behavior of the transformation function.

Keywords: Extended transformation, Schrodinger Green's function equation, bound state and scattering state solutions, non-power law potential, Exactly solvable potential.

1. Introduction

In non-relativistic quantum mechanics maximum information of a quantum system (QS) can had when one knows the exact solutions of the corresponding Schrodinger equation. In the recent years considerable interest has been made, in the study of exactly solvable quantum mechanical potentials of physical interest. Various methods are used in the calculation of exact analytic solutions [1,2,3,4,5,6,7,8,9,10,11]. Instead of the conventional methods we present a new method for constructing the Green functions systematically. Green's function (GF) technique offers a powerful and convenient method of solving differential equations. Obtaining the energy dependent GF of the Schrodinger equation as yields the eigenvalues and normalized eigenfunctions. Besides it also provides scattering state solutions under different set of boundary condition. The present work dealt with an interesting general relation between the energy dependent GF of different physical problems. The connection takes a remarkably compact form when expressed in terms of the radial part of the GF for spherically symmetric potentials. Such a relation was first obtained by Steiner [12,13] in the course of his work on radial path integral. S.S Vasan, M. Seetharaman and K. Raghunathan [14] organized a general connection between the green's functions for different potentials. But our aim and objectives are quite different. Our principal aim here is to demonstrate general relation between the GFs for different QSs of physical relevance directly from the differential equation satisfied by the GF and it is expressed in terms of the radial part of the for spherically symmetric potentials. For this purpose we have to made a transformation method called extended transformation (ET) method [15,16,17,18, 19, 20, 21] within the framework of GF technique, in any chosen D-dimensional space. ET consist of a coordinate transformation followed by functional transformation and a plausible ansatze. We start from an analytically solved potentials of a given QS and transformed it to a class of exactly solvable potentials for new QSs.

The paper is structured as follows: In section 2. We review briefly the ET method satisfied by Schrodinger GF equation. In sub section 3.1, 3.2 and 3.3 application of the transformation method to construct three classes of exactly solved QSs by taking three working potential. Conclusions and findings of our results are discussed in section 4.

2. Formalism:

The radial part of the s-wave Schrodinger equation for the potential $V_A(r)$ in D_A-dimensional Euclidean space ($\hbar = 1 = 2m$):

$$\left[\partial_r^2 + \frac{D_A-1}{r}\partial_r + E_n^A - V_A(r)\right] G_A(r, r_0; E_n^A) = \frac{\delta(r-r_0)}{r_0^{D_A-1}} \tag{1}$$

where r is a dimensionless modulus of radius vector in the D_A-dimensional space.

The corresponding integral equation is:

$$\varphi_A(r) = \int G_A(r, r_0; E_A)(E_n^A - V_A(r))\varphi_A(r_0) r_0^{D_A-1} d\, r_0 \tag{2}$$

The normalized eigenfunctions $\varphi_A(r)$ and energy eigenvalues E_n^A are known for the given potential $V_A(r)$.

The completeness of the set of energy eigenfunctions allows us to have eigenfunctions expansion of the GF:

$$G_A(r, r_0; E_n^A) = \sum_{n=0}^{\infty} \frac{\varphi_A(r)\varphi_A^*(r_0)}{E - E_n^A - i\varepsilon} \tag{3}$$

from which we read off the analytical form of the wave function of the solved quantum system.

We now invoke a coordinate transformation $r \to g_B(r),\ r_0 \to g_B(r_0)$ and a functional transformation

$$G_B(r, r_0; E_n^B) = f_B^{-1}(r) G_A(g_B(r), g_B(r_0); E_n^A) f_B^{-1}(r_0) \tag{4}$$

where the transformation functions $g_B(r)$ and $g_B(r_0)$ and the modulated amplitudes $f_B(r)$ and $f_B(r_0)$ have get specified within the framework of ET.

The transformed B-QS after implementing ET on the already solved A-QS takes the form:

$$\left[\partial_r^2 + \left(\frac{d}{dr} ln \frac{f_B^2(r) g_B^{D_A-1}(r)}{g'_B(r)}\right)\partial_r + \left(\frac{d}{dr} ln f_B(r)\right)\left(\frac{d}{dr} ln \frac{f'_B g_B^{D_A-1}(r)}{g'_B(r)}\right) + g'^2_B(E_n^A - V_A(g_B))\right] G_B(r, r_0; E_n^A) = g'^2_B f_B^{-1}(r) \frac{\delta(g_B(r) - g_B(r_0))}{g_B^{D_A-1}(r_0)} f_B^{-1}(r_0) \tag{5}$$

The prime denotes differentiation with respect to the variable r.

The dimension of the Euclidean spaces of the transformed QS, henceforth called the B-QS, can be chosen arbitrarily, let it be denoted by D_B.

This requires to put, the coefficient $\frac{d}{dr}$

$$\frac{d}{dr} ln \frac{f_B^2(r) g_B^{D_A-1}(r)}{g'_B(r)} = \frac{D_B-1}{r} = \ln r^{D_B-1} \tag{6}$$

From equations (4) and (6), we obtain

$$G_B(r, r_0; E_n^B) = g'^{-\frac{1}{2}}_B(r) g_B^{\frac{D_A-1}{2}}(r) r^{-\frac{D_B-1}{2}} G_A\left(g_B(r), g_B(r_0); E_n^A\right) g'^{-\frac{1}{2}}_B(r_0) g_B^{\frac{D_A-1}{2}}(r_0) r_0^{-\frac{D_B-1}{2}}(r_0) \tag{7}$$

The transformation function $g_B(r)$ is at least three times differentiable.

The corresponding second order equation for B-QS in D_B-dimensional space becomes:

$$\left[\partial_r^2 + \frac{D_B-1}{r}\partial_r + \frac{1}{2}\{g_B, r\} + g'^2_B\left(E_n^A - V_A\left(g_B(r)\right)\right)\right] G_B(r, r_0; E_n^B) = \frac{\delta(r-r_0)}{r_0^{D_B-1}} \tag{8}$$

with Schwatzain derivative symbol

$$\{g_B, r\} = \frac{g_B'''}{g_B'} - \frac{3}{2}\left(\frac{g_B''}{g_B'}\right)^2 \tag{9}$$

In case multiterm A-QS to implement ET we have to select one or more term(s) of $V_A\left(g_B(r)\right)$ as working potential (WP) and is designated by $V_A^W(g_B(r))$.

In order to mould equation (8) to the standard Schrodinger GF equation form, following plausible ansatze have to be made which are an integral part of the transformation method:

$$g'^2_B V_A^W\left(g_B(r)\right) = -E_n^B \tag{10}$$

$$g'^2_B E_n^A = -V_B^{(1)}(r) \tag{11}$$

$$g'^2_B\left(V_A\left(g_B(r)\right) - V_A^W\left(g_B(r)\right)\right) = V_B^{(2)}(r) \tag{12}$$

$$\frac{1}{2}\{g_B, r\} = -V_B^{(3)}(r) \tag{13}$$

These ansatze leads the B-QS potential

$$V_B(r) = V_B^{(1)}(r) + V_B^{(2)}(r) + V_B^{(3)}(r) \tag{14}$$

Further, it is to be noted that, the equation (10) specifies the functional form of the transformation function $g_B(r)$.

The familiar radial Schrodinger GF equation for B-QS in D_B-dimensional spaces takes the form:

$$\left[\partial_r^2 + \frac{D_B-1}{r}\partial_r + E_n^B - V_B(r)\right] G_B(r, r_0; E_n^B) = \frac{\delta(r-r_0)}{r_0^{D_B-1}} \tag{15}$$

3. Construction of Exactly solvable QSs from Generalized Hulthen Potential

To illustrate an application of our ET formalism we have considered the Generalized Hulthen system whose exact solution exists only for $l = 0$ case only. The Generalized Hulthen system is a true representation of non-power law potential. Let it be denoted as our A-system.

The radial Schrodinger GF equation for this potential, in three dimensional space is:

$$\left[\partial_r^2 + E_n^A - \left(\frac{\alpha^2\mu^2}{1-e^{-2\alpha r}} - (\lambda^2+\alpha^2)\frac{e^{-2\alpha r}}{1-e^{-2\alpha r}} - \frac{3}{4}\alpha^2\frac{e^{-4\alpha r}}{(1-e^{-2\alpha r})^2}\right)\right] G_B(r, r_0; E_n^A) = \frac{\delta(r-r_0)}{r_0^2} \quad (16)$$

where

$$V_A(r) = \frac{\alpha^2\mu^2}{1-e^{-2\alpha r}} - (\lambda^2+\alpha^2)\frac{e^{-2\alpha r}}{1-e^{-2\alpha r}} - \frac{3}{4}\alpha^2\frac{e^{-4\alpha r}}{(1-e^{-2\alpha r})^2} \quad (17)$$

is the Generalized Hulthen potential [15]. The energy eigenvalues are

$$E_n^A = \frac{1}{4}\frac{\lambda^2}{(n+\mu)(n+\mu+1)} \quad (18)$$

The constraint equation for the parameters is

$$(n+\mu)(n+\mu+1) - \frac{\lambda^2}{\alpha^2} = 0 \quad (19)$$

where the angular momentum quantum number and the magnetic quantum number

$$\mu = \frac{1}{2}\left(1+\frac{4\lambda^2}{\alpha^2}\right) - \left(n+\frac{1}{2}\right) > 0 \quad (20)$$

The number of bound states is restricted by the integral part of $(n+\mu)$.

The exact s-wave energy eigenfunctions are given as

$$\varphi_A(r) = N_A r^{-1}(1-e^{-2\alpha r})^{1/4} P_{n+\mu}^{\mu}\{(1-e^{-2\alpha r})^{1/2}\} \quad (21)$$

where N_A is the normalization constant.

The A-QS potential is a three term potential. In case of multiterm potential the transformation procedure may be applied repeatedly, by selecting working potential (WP) differently to generate a variety of solved quantum system (QS). The number of solvable QS in principle, being equal to the number of ways (2^3-1) in which the WP may be chosen when the original QS is governed by a QS with 3-term potential. We however restrict ourselves to taking one term WP. Two or three term WP as they offer following practical difficulties: the indefinite integral specifying the transformation function $g_B(r)$ cannot be evaluated analytically in most of the cases and even if such integrals are found they are the form $F(g_B(r)) = r + C$ and the analytical inverse function $F^{-1}(g_B(r))$ cannot be found.

3.1 Choosing the first term from equation (17) as WP i.e.,

$$V_A^{(W)}(g_B(r)) = \frac{\alpha^2\mu^2}{1-e^{-2\alpha g_B(r)}} \quad (22)$$

The appended ansatze (10), (11) and (12) becomes

$$g_B'^2\frac{\alpha^2\mu^2}{1-e^{-2\alpha g_B(r)}} = -E_n^B \quad (23)$$

$$g_B'^2\frac{1}{4}\frac{\lambda^2}{(n+\mu)(n+\mu+1)} = -V_B^{(1)}(r) \quad (24)$$

$$g_B^{'2}\left(-(\lambda^2+\alpha^2)\frac{e^{-2\alpha g_B(r)}}{1-e^{-2\alpha g_B(r)}}-\frac{3}{4}\alpha^2\frac{e^{-4\alpha g_B(r)}}{(1-e^{-2\alpha g_B(r)})^2}\right)=V_B^{(2)}(r) \qquad (25)$$

The functional form of $g_B(r)$ is obtained from equation (23) by integration as:

$$g_B(r)=\frac{1}{\alpha}\ln\cos h\,\alpha k_B r \qquad (26)$$

where

$$k_B=\sqrt{\frac{E_n^B}{\alpha^2\mu^2}} \qquad (27)$$

The integration constant is set equal to zero, which attributes the local property $g_B(0)=0$. Equations (24) and (26) yields:

$$V_B^{(1)}(r)=-C_B^2\tan h^2\alpha\, k_B r \qquad (28)$$

We put

$$k_B^2(-E_n^A)=C_B^2 \qquad (29)$$

where C_B^2 is the characteristic constant of the newly constructed B-QS, obtained from the transformation of A-QS. Equation (29) subsequently provides us the B-system energy eigenvalues.

Equation (25) and (26) lead to

$$V_B^{(2)}(r)=\frac{1}{4}\alpha^2k_B^2\tan h^2\alpha\, k_B r-(\lambda^2+\alpha^2)k_B^2\sec h^2\alpha\, k_B r-\frac{3}{4}\alpha^2k_B^2\frac{\sec h^4\alpha\, k_B r}{\tan h^2\alpha\, k_B r} \qquad (30)$$

Equation (13) and (26) give

$$V_B^{(3)}(r)=\alpha^2k_B^2\sec h^2\alpha\, k_B r+\frac{3}{4}\alpha^2k_B^2\frac{\sec h^4\alpha\, k_B r}{\tan h^2\alpha\, k_B r} \qquad (31)$$

Then from equation (14) the B-QS potential is found as

$$V_B(r)=-C_B^2\tan h^2\alpha\, k_B r \qquad (32)$$

The potential is similar to the Rosen-Morse potential.

The energy eigenvalues of the $V_B(r)$ given by the equation (19) and (29) are

$$E_n^B=-\beta^2\left[\frac{1}{2}\left(1+\frac{4C_B^2}{\alpha^2}\right)-\left(n+\frac{1}{2}\right)\right] \qquad (33)$$

where $\beta=\sqrt{\frac{-E_n^B}{\mu}}$.

Utilizing equations (3) and (4) we obtain the eigenfunction expansion of B-QS as:

$$G_B(r,r_0;E_n^B)=\sum_{n=0}^{\infty}\frac{r^{-1}P_{n+\mu}^{\mu}(\tan\alpha r)r_0^{-1}P_{n+\mu}^{\mu}(\tan\alpha r_0)}{E+\beta^2\left[\frac{1}{2}\left(1+\frac{4C_B^2}{\alpha^2}\right)-\left(n+\frac{1}{2}\right)\right]-i\epsilon} \qquad (34)$$

From equation (34) the new radial wave function is read off in three dimensional space which is:

$$\varphi_B(r) = N_B r^{-1} P^{\mu}_{n+\mu}(\tan \alpha r) \quad (35)$$

The familiar Schrodinger GF equation in 3-dimensional Euclidean space is found as:

$$\left[\partial_r^2 + \frac{2}{r}\partial_r - \beta^2\left[\frac{1}{2}\left(1+\frac{4C_B^2}{\alpha^2}\right) - \left(n+\frac{1}{2}\right)\right] + C_B^2 \tan h^2 \alpha\, k_B r\right] G_B(r, r_0; E_n^B) = \frac{\delta(r-r_0)}{r_0^2} \quad (36)$$

We now consider the Green's function for the positive value of E_B. In this case we have replaced E_B in terms of 'k' ($E_B = k^2 > 0$), the asymptotic wave number in atomic units and the energy spectrum in the entire real continuum.

The relevant part of the transformation method now

$$g_B^{'2} \frac{\alpha^2\mu^2}{1 - e^{-2\alpha g_B(r)}} = -k^2 \quad (37)$$

We have found the transformation function for the scattering state by integrating equation (37) as:

$$g_B(r) = \frac{1}{\alpha} \ln cos\, h\left(\frac{ik}{\alpha\mu} r\right) \quad (38)$$

Utilizing equations (13), (24), (25) and (38) we found the scattered potential of B-QS as:

$$V_B^{scatt}(r) = \left(\frac{\lambda k}{\alpha^2 \mu}\right)^2 \sec h^2 \frac{ik}{\alpha\mu} r \quad (39)$$

Using the orthohogonality and completeness relations of the eigenfunctions $\varphi_A(g_B(r))$, we can write the analogue of equation (34)

$$G_B(r, r_0; k^2) = \int_0^{\infty} \frac{dE}{E - k^2 - i\epsilon} f_B^{-1}(r) \varphi_A\big(g_B(r)\big) \varphi_A^*\big(g_B(r_0)\big) f_B^{-1}(r_0) \quad (40)$$

To determine the B-Qs wave function belonging to continuum we use the symbolic identity:

$$\lim_{r\to\infty} \frac{1}{(E-k^2) \pm i\epsilon} = P\frac{1}{E-k^2} \mp i\pi\delta(E-k^2) \quad (41)$$

The imaginary part of equation (41) can be determined through equation (41) which is:

$$G_B(r, r_0; k^2) = \pi \int_0^{\infty} dE\ \delta(E-k^2)\, f_B^{-1}(r) \varphi_A\big(g_B(r)\big) \varphi_A^*\big(g_B(r_0)\big) f_B^{-1}(r_0) \quad (42)$$

where the scattering state wave function is

$$\varphi_B^{scatt}(r) = N_B r^{-1} P_L^M\left(-tan\frac{ik}{\alpha\mu} r\right) \quad (43)$$

3.2 Choosing the second term from equation (17) as WP i.e.,

$$V_A^{(W)}\big(g_C(r)\big) = -(\lambda^2 + \alpha^2) \frac{e^{-2\alpha g_C(r)}}{1 - e^{-2\alpha g_C(r)}} \quad (44)$$

Application of ET on A-QS for the WP cited above we can constructed another new QS, which is designated by C-QS, by the above procedure.

Utilizing equation (10), we have found the transformation function

$$g_C(r) = \frac{1}{\alpha} \ln \sec \alpha k_C r \tag{45}$$

where

$$k_C = \sqrt{\frac{E_n^C}{\lambda^2 + \alpha^2}} \tag{46}$$

Equations (11) and (45) yield

$$V_C^{(1)}(r) = k_C^2(-E_n^A)\alpha k_C r \tag{47}$$

we put

$$k_C^2(-E_n^A) = C_C^2 \tag{48}$$

where $C_C^2 = \gamma\ (say)$ is the characteristic constant of the constructed C-QS obtained from the transformation of A-QS.

Equations (12) and (45) lead to

$$V_C^{(2)}(r) = \frac{1}{4}\alpha^2 k_C^2 tan^2 \alpha k_C r + \alpha^2 \mu^2 sec^2 \alpha k_C r - \frac{3}{4}\alpha^2 k_C^2 cot^2 \alpha k_C r \tag{49}$$

Equation (13) and (45) give

$$V_C^{(3)}(r) = \alpha^2 k_C^2 sec^2 \alpha k_C r + \frac{3}{4}\alpha^2 k_C^2 cot^2 \alpha k_C r - \frac{3}{4}\alpha^2 k_C^2 tan^2 \alpha k_C r + \frac{3}{2}\alpha^2 k_C^2 \tag{50}$$

Then from equation (14) the C-QS potential is found as

$$V_C(r) = \alpha^2 \mu^2 sec^2 \alpha k_C r - \frac{1}{4}\alpha^2 k_C^2 tan^2 \alpha k_C r + \frac{1}{2}\alpha^2 k_C^2 \tag{51}$$

The quantized energy eigenvalues of C-QS are obtained as:

$$E_n^C = 4\gamma\{(n+\mu)(n+\mu+1)+1\} \tag{52}$$

Utilizing equations (3) and (4) we obtain the eigenfunction expansion of C-QS as:

$$G_C(r, r_0; E_n^C) = \sum_{n=0}^{\infty} \frac{r^{-1} cos^{1/2}(2\gamma r) P_{n+\mu}^{\mu}(\sin 2\gamma r) r_0^{-1} cos^{1/2}(2\gamma r) P_{n+\mu}^{\mu}(\sin 2\gamma\, r_0)}{E - 4\gamma\{(n+\mu)(n+\mu+1)+1\} - i\epsilon} \tag{53}$$

Which allow us to obtain the bound state eigenfunction in three dimensional space as:

$$\varphi_C(r) = N_c r^{-1} cos^{\frac{1}{2}}(2\gamma r) P_{n+\mu}^{\mu}(\sin 2\gamma r) \tag{54}$$

3.3 Choosing the third term from equation (17) as WP

By the application of ET on A-QS as obtained from equation (17), taking third term as working potential, we can construct another new QS, which is designated by D-QS, by the above procedure. The working potential is

$$V_A^{(W)}(g_D(r)) = -\frac{3}{4}\alpha^2 \frac{e^{-4\alpha g_D(r)}}{(1 - e^{-2\alpha g_D(r)})^2} \tag{55}$$

The transformation function is found as

$$g_D(r) = -\frac{1}{2\alpha} ln(1 - e^{2\alpha k_D r}) \qquad (56)$$

where

$$k_D = \sqrt{\frac{4}{3}\frac{E_D}{\alpha^2}} \qquad (57)$$

The form of the newly constructed D-QS potential comes out to be

$$V_D(r) = -(\lambda^2 + \alpha^2)k_D^2 \frac{e^{-2\alpha k_D r}}{1 - e^{-2\alpha k_D r}} + \alpha^2\mu^2 k_D^2 \frac{e^{-2\alpha k_D r}}{1 - e^{-2\alpha k_D r}} - \alpha^2 k_D^2 \frac{e^{-4\alpha k_D r}}{(1 - e^{-2\alpha k_D r})^2} \qquad (58)$$

The generated potential is designated as generalized Hulthen potential, i.e., after the application of ET on A-QS multi-term potential, one term reverse back to the original potential.

For the potential $V_D(r)$, we have found constant energy eigenvalues as:

$$E_D = \frac{3}{4}\alpha^2 k_D^2 \qquad (59)$$

The eigenfunction expanded form of GF is now

$$G_D(r, r_0; E_D) = \sum_{n=0}^{\infty} \frac{r^{-1} sinh^{1/2}(\alpha k_D r) P_{n+\mu}^{\mu}(\cos h(\alpha k_D r) + \sin h(\alpha k_D r))\, r_0^{-1} sinh^{1/2}(\alpha k_D r) P_{n+\mu}^{\mu}(\cos h(\alpha k_D r_0) + \sin h(\alpha k_D r_0))}{E - \frac{3}{4}\alpha^2 k_D^2 - i\epsilon} \qquad (60)$$

which allows us to obtain the bound state eigenfunction $\varphi_D(r)$ as

$$\varphi_D(r) = N_D r^{-1} sinh^{\frac{1}{2}}(\alpha k_D r) P_{n+\mu}^{\mu}(\cos h(\alpha k_D r) + \sin h(\alpha k_D r)) \qquad (61)$$

To avoid repetition we have give generated scattered potentials, transformation functions and scattering state wave functions are in tabular form.

Table-1: List of scattered potentials obtained from different A-QS working potentials.

Sl. No	Working potential	Constructed potential
1	$\frac{\alpha^2\mu^2}{1 - e^{-2\alpha g_B(r)}}$	$V_B^{scatt} = \left(\frac{\lambda k_1}{\alpha^2\mu}\right)^2 \sec h^2\left(\frac{ik_1}{\alpha\mu} r\right)$
2	$-(\lambda^2 + \alpha^2)\frac{e^{-2\alpha g_C(r)}}{1 - e^{-2\alpha g_C(r)}}$	$V_C^{scatt} = k_2^2\mu^2 sec^2(ik_2 r) + \frac{1}{4}k_2^2 tan^2(ik_2 r) - \frac{1}{2}k_2^2$
3	$-\frac{3}{4}\alpha^2 \frac{e^{-4\alpha g_D(r)}}{(1 - e^{-2\alpha g_D(r)})^2}$	$V_D^{scatt} = (\lambda^2 + \alpha^2)k_3^2 \frac{e^{2ik_3 r}}{1 - e^{2ik_3 r}} + \alpha^2\mu^2 k_3^2 \frac{e^{e^{2ik_3 r}}}{1 - e^{e^{2ik_3 r}}} - \alpha^2 k_3^2 \frac{e^{4ik_3 r}}{\left(1 - e^{e^{2ik_3 r}}\right)^2}$

Table-2: List of transformation functions and scattered wave functions for the WP given in table-1

Sl. No.	Transformation functions	Scattered wave functions
1	$\frac{1}{\alpha}\ln \cos h\left(\frac{ik_1}{\alpha\mu}r\right)$	$N_B r^{-1} P_L^M\left(-tan\frac{ik_1}{\alpha\mu}r\right)$
2	$\frac{1}{\alpha}\ln cos\,(ik_2 r)$	$N_C r^{-1} cos^{\frac{1}{2}}(ik_2 r)P_L^M(sinik_2 r)$
3	$-\frac{1}{2\alpha}ln\left(1-e^{2ik_3 r}\right)$	$N_D r^{-1}\left(e^{-ik_3 r}-e^{ik_3 r}\right)P_L^M(e^{ik_3 r})$

4. Discussion and conclusion

Using the ET method, we have considered the generation of exact analytic solutions of the Schrodinger equation for the GF in the form of bound states and scattering states. In this paper we have presented three new classes of exactly solvable quantum systems in non relativistic quantum mechanics in three dimensional Euclidean spaces, within the framework of GF technique. The main transformation relation of GF is given by equation (4), which satisfies the transformed Schrodinger GF for the new potential $V_B(r)$ given by (15). The dimension of the Euclidean space of the transformed QS can be chosen through equation (6). Application of the ET method to the A-QS potential $V_A(r)=\frac{\alpha^2\mu^2}{1-e^{-2\alpha r}}-(\lambda^2+\alpha^2)\frac{e^{-2\alpha r}}{1-e^{-2\alpha r}}-\frac{3}{4}\alpha^2\frac{e^{-4\alpha r}}{(1-e^{-2\alpha r})^2}$, known as generalized Hulthen potential. The selection of working potential can be made, in principle, in $(2^3-1=7)$ for three term potential. We have restricted ourselves to taking one term WP. Two and more than two terms WP offers some practical difficulties. Choosing the WP $V_A^{(W)}(g_B(r))=\frac{\alpha^2\mu^2}{1-e^{-2\alpha g_B(r)}}$ leads to B-QS potential $V_B(r)=-C_B^2\tan h^2\alpha\, k_B r$, which is a scaled version of the Rosen-Morse QS. Taking the WP $V_A^{(W)}(g_C(r))=-(\lambda^2+\alpha^2)\frac{e^{-2\alpha g_C(r)}}{1-e^{-2\alpha g_C(r)}}$ we have generated C-QS potential $V_C(r)=\alpha^2\mu^2 sec^2\alpha k_C r-\frac{1}{4}\alpha^2 k_C^2 tan^2\alpha k_C r+\frac{1}{2}\alpha^2 k_C^2$ and when we taken $V_A^{(W)}(g_D(r))=-\frac{3}{4}\alpha^2\frac{e^{-4\alpha g_D(r)}}{\left(1-e^{-2\alpha g_D(r)}\right)^2}$ as WP, Et generates D-QS potential $V_D(r)=-(\lambda^2+\alpha^2)k_D^2\frac{e^{-2\alpha k_D r}}{1-e^{-2\alpha k_D r}}+\alpha^2\mu^2 k_D^2\frac{e^{-2\alpha k_D r}}{1-e^{-2\alpha k_D r}}-\alpha^2 k_D^2\frac{e^{-4\alpha k_D r}}{\left(1-e^{-2\alpha k_D r}\right)^2}$, the potential reverse back to the original potential. Green's function can handle both bound state and scattering state with some adjustment of the parameters. The scattering states solutions are given in table-1and table-2.

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