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P-Posinormal and (p,k)-Quasi Posinormal Composite Multiplication Operators on the Complex Hilbert Space

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ABSTRACT

In this paper, the condition under which composite multiplication operators on $L^2(\mu)$-space become the characterizations of posinormal and quasi-posinormal composite multiplication operators have been given. Also p-posinormal, k-quasi-posinormal and (p,k) -quasi-posinormal operators have been obtained in terms of radon-nikodym derivative f_0 .

Keywords: Composite multiplication operator, conditional expectation, p-posinormal, k- Quasi posinormal, (p,k) -quasi-posinormal.

Mathematics Subject Classification 2010: 47B33, 47B20, 46C05.

1. INTRODUCTION

Let (X,Σ,μ) be a σ-finite measure space. Then a mapping T from X into X is said to be a measurable transformation if $T^{-1}(E)\in\Sigma$ for every $E\in\Sigma$ .A measurable transformation T is said to be non-singular if $\mu(T^{-1}(E))=0$ whenever $\mu(E)=0$. If T is non-singular then the measure μT^{-1} defined as $\mu T^{-1}(E)=\mu(T^{-1}(E))$ for every E in Σ , is an absolutely continuous measure on Σ with respect to μ.Since μ is a σ-finite measure, then by the Radon-Nikodym theorem, there exists a non-negative function f_0 in $L^1(\mu)$ such that $\mu T^{-1}(E)=\int_E f_0\, d\mu$ for every $E\in\Sigma$. The function f_0 is called the Radon-Nikodym derivative of μT^{-1} with respect to μ.

Every non- singular measurable transformation T from X into itself induces a linear transformation C_T on $L^p(\mu)$ defined as $C_T f = f\circ T$ for every f in $L^p(\mu)$. In case C_T is continuous from $L^p(\mu)$ into itself, then it is called a composition operator on $L^p(\mu)$ induced by T. We restrict our study of the composition operators on $L^2(\mu)$ which has Hilbert space structure. If u is an essentially bounded complex-valued measurable function on X, then the mapping M_u on $L^2(\mu)$ defined by $M_u f = u\cdot f$, is a continuous operator with range in $L^2(\mu)$. The operator M_u is known as the multiplication operator induced by u.

A composite multiplication operator is linear transformation acting on a set of complex valued Σ measurable functions f of the form

$$M_{u,T}(f) = C_T M_u(f) = u \circ T \; f \circ T$$

Where u is a complex valued, Σ measurable function. In case $u = 1$ almost everywhere, $M_{u,T}$ becomes a composition operator, denoted by C_T.

In the study considered is the using conditional expectation of composite multiplication operator on L^2-spaces. For each $f \in L^p(X,\Sigma,\mu)$, $1 \le p \le \infty$, there exists an unique $T^{-1}(\Sigma)$-measurable function $E(f)$ such that

$$\int_A g f \, d\mu = \int_A g E(f) \, d\mu$$

for every $T^{-1}(\Sigma)$-measurable function g, for which the left integral exists. The function $E(f)$ is called the conditional expectation of f with respect to the subalgebra $T^{-1}(\Sigma)$. As an operator of $L^p(\mu)$, E is the projection onto the closure of range of T and E is the identity on $L^p(\mu)$, $p \ge 1$ if and only if $T^{-1}(\Sigma) = \Sigma$. Detailed discussion of E is found in [1-4].

1.1 Posinormal

An operator $A \in B(H)$, A is said to be posinormal if $AA^* \le c^2 A^*A$ for some $c > 0$.

1.2 p-posinormal

An operator $A \in B(H)$, A is said to be p-posinormal if $(AA^*)^p \le c^2 (A^*A)^p$ for some $c > 0$, $0 < p \le 1$.

1.3 Quasi-posinormal

An operator $A \in B(H)$, A is said to be quasi-posinormal if $A^*(AA^*)A \le c^2 A^{*2}A^2$ for some $c > 0$.

1.4 k-Quasi-posinormal

An operator $A \in B(H)$, A is said to be k-quasi-posinormal if $A^{*k}(AA^*)A^k \le c^2 A^{*(k+1)}A^{(k+1)}$ for some $c > 0$, where k is a positive integer.

1.5 (p,k) -Quasi-posinormal

An operator $A \in B(H)$, A is said to be (p,k) -quasi-posinormal if

$A^{*k}(AA^*)^p A^k \le c^2 A^{*k}(A^*A)^p A^k$

for some $c > 0$, $0 < p \le 1$.

2. RELATED WORK IN THE FIELD

The study of weighted composition operators on L^2 spaces was initiated by R.K.Singh and D.C.Kumar [5]. During the last thirty years, several authors have studied the properties of various classes of weighted composition operator. Boundedness of the composition operators in $L^p(\Sigma), (1 \le p < \infty)$ spaces, where the measure spaces are σ-finite, appeared already in [6]. Also boundedness of weighted operators on $C(X,E)$ has been studied in [7]. Recently S.Senthil, P.Thangaraju, Nithya M, Surya devi B and D.C.Kumar, have proved several theorems on n-normal , n-quasi-normal, k-paranormal, and (n,k) paranormal of composite multiplication operators on L^2 spaces [8-12]. In this paper we investigate composite multiplication operators on $L^2(\mu)$-space become Quasi-P-Normal operators and n-Power class Q operator have been obtained in terms of radon-nikodym derivative f_0 .

3. CHARACTERIZATION ON COMPOSITE MULTIPLICATION OF POSINORMAL AND QUASI-POSINORMAL OPERATORS ON L^2-SPACE

3.1 Proposition:

Let the composite multiplication operator $M_{u,T} \in B(L^2(\mu))$. Then for $u \geq 0$

(i) $M^*_{u,T} M_{u,T} f = u^2 f_0 f$

(ii) $M_{u,T} M^*_{u,T} f = u^2 \circ T \cdot f_0 \circ T \cdot E(f)$

(iii) $M^n_{u,T}(f) = (C_T M_u)^n (f) = u_n (f \circ T^n)$, $u_n = u \circ T . u \circ T^2 . u \circ T^3 ....u \circ T^n$

(iv) $M^*_{u,T} f = u f_0 \cdot E(f) \circ T^{-1}$

(v) $M^{*^n}_{u,T} f = u f_0 \cdot E(u f_0) \circ T^{-(n-1)} \cdot E(f) \circ T^{-n}$

where $E(u f_0) \circ T^{-(n-1)} = E(u f_0) \circ T^{-1} \cdot E(u f_0) \circ T^{-2} .... E(u f_0) \circ T^{-(n-1)}$

Theorem 3.2

Let the composite multiplication operator $M_{u,T} \in B(L^2(\mu))$. Then $M_{u,T}$ is posinormal if and only if $u^2 \circ T \cdot f_0 \circ T \, E(f) \leq C^2 u^2 f_0 f$ almost everywhere, for all $C \geq 0$

Proof:

Suppose $M_{u,T}$ is posinormal. Then $M_{u,T} M^*_{u,T} \leq C^2 \, M^*_{u,T} M_{u,T}$ for all $C \geq 0$.

This implies that

$\langle (M_{u,T} M^*_{u,T} - C^2 M^*_{u,T} M_{u,T}) f, f \rangle \leq 0$ for all $f \in L^2(\mu)$

Since $M_{u,T} M^*_{u,T} f = u^2 \circ T \cdot f_0 \circ T \cdot E(f)$ and $M^*_{u,T} M_{u,T} f = u^2 f_0 f$ $u \geq 0$ we have

$\int_E \{ (u^2 \circ T \cdot f_0 \circ T \cdot E(f) - C^2 \, u^2 f_0 f \} d\mu \leq 0$ for every $E \in \Sigma$.

$\Leftrightarrow u^2 \circ T \cdot f_0 \circ T \, E(f) \leq C^2 u^2 f_0 f$ almost everywhere, for all $C \geq 0$

Corollary 3.3

The composition operator C_T on $B(L^2(\mu))$ is posinormal if and only if

$f_0 \circ T \, E(f) \leq C^2 f_0 f$ almost everywhere, for all $C \geq 0$

Proof:

The proof is obtained from Theorem 3.2 by putting $u = 1$.

Theorem 3.4

Let the composite multiplication operator $M_{u,T} \in B(L^2(\mu))$. Then $M_{u,T}$ is quasi-posinormal if and only if $u^2 f_0 \leq C^2 E(u^2 f_0) \circ T^{-1}$ almost everywhere, for all $C \geq 0$.

Proof:

Suppose $M_{u,T}$ is quasi-posinormal. Then $(M^*_{u,T}M_{u,T})^2 \le C^2 M^{*^2}_{u,T} M^2_{u,T}$ for all $C \ge 0$.

This implies that

$$\left\langle ((M^*_{u,T}M_{u,T})^2 - C^2 M^{*^2}_{u,T} M^2_{u,T}) f , f \right\rangle \le 0 \text{ for all } f \in L^2(\mu)$$

Since $M^*_{u,T} M_{u,T} f = u^2 f_0 f$ and $M^{*^2}_{u,T} M^2_{u,T} f = u^2 f_0 E(u^2 f_0) \circ T^{-1} f$ $u \ge 0$, we have

$$\int_E \left\{ u^4 f_0^2 f - C^2 u^2 f_0 E(u^2 f_0) \circ T^{-1} f \right\} d\mu \le 0 \text{ for every } E \in \Sigma \quad .$$

$\Leftrightarrow$ $u^2 f_0 \le C^2 E(u^2 f_0) \circ T^{-1}$ almost everywhere, for all $C \ge 0$

Corollary 3.5

The composition operator C_T on $B(L^2(\mu))$ is quasi-posinormal if and only if

$f_0 \le C^2 E(f_0) \circ T^{-1}$ almost everywhere, for all $C \ge 0$

Proof:

The proof is obtained from Theorem 3.4 by putting $u = 1$.

Theorem 3.6

Let the composite multiplication operator $M_{u,T} \in B(L^2(\mu))$. Then $M^*_{u,T}$ is quai-posinormal if and only if $u^4 \circ T\, f_0^2 \circ T\, E(f) \le C^2 u \circ T\, u^2 \circ T^2\, f_0 \circ T^2\, E(uf_0) \circ T\, E(f)$ almost everywhere, for all $C \ge 0$.

Proof:

Suppose $M^*_{u,T}$ is quai-posinormal. Then $(M_{u,T}M^*_{u,T})^2 \le C^2 M^2_{u,T} M^{*^2}_{u,T}$ for all $C \ge 0$.

This implies that

$$\left\langle ((M_{u,T}M^*_{u,T})^2 - C^2 M^2_{u,T} M^{*^2}_{u,T}) f , f \right\rangle \le 0 \text{ for all } f \in L^2(\mu)$$

Since $M_{u,T} M^*_{u,T} f = u^2 \circ T \cdot f_0 \circ T \cdot E(f)$ and

$M^2_{u,T} M^{*^2}_{u,T} f = u \circ T \cdot u^2 \circ T^2 \cdot f_0 \circ T^2 \cdot E(uf_0) \circ T \cdot E(f)$, $u \ge 0$ we have

$$\int_E \left\{ (u^2 \circ T \cdot f_0 \circ T \cdot E(f))^2 - C^2 u \circ T \cdot u^2 \circ T^2 \cdot f_0 \circ T^2 \cdot E(uf_0) \circ T \cdot E(f) \right\} d\mu \le 0 \text{ for every } E \in \Sigma \quad .$$

$\Leftrightarrow$ $u^4 \circ T\, f_0^2 \circ T\, E(f) \le C^2 u \circ T\, u^2 \circ T^2\, f_0 \circ T^2\, E(uf_0) \circ T\, E(f)$ almost everywhere, for all $C \ge 0$

Corollary 3.7

The composition operator C_T^* on $B(L^2(\mu))$ is quasi-posinormal if and only if

$f_0^2 \circ T\, E(f) \le C^2 f_0 \circ T^2\, E(f_0) \circ T\, E(f)$ almost everywhere, for all $C \ge 0$

Proof:

The proof is obtained from Theorem 3.6 by putting $u = 1$.

4. CHARACTERIZATION ON COMPOSITE MULTIPLICATION OF P-POSINORMAL AND K-QUASI-POSINORMAL OPERATORS ON L^2-SPACE

Theorem 4.1

Let the composite multiplication operator $M_{u,T} \in B(L^2(\mu))$. Then $M_{u,T}$ is p-posinormal if and only if $u^{2p} \circ T\, f_0{}^p \circ T\, E(f) \leq C^2\, u^{2p} f_0{}^p f$ almost everywhere, for all $C \geq 0$

Proof:

Suppose $M_{u,T}$ is p-posinormal. Then $(M_{u,T} M^*{}_{u,T})^p \leq C^2\, (M^*{}_{u,T} M_{u,T})^p$ for all $C \geq 0$.

This implies that

$$\left\langle ((M_{u,T} M^*{}_{u,T})^p - C^2\, (M^*{}_{u,T} M_{u,T})^p) f,\ f \right\rangle \leq 0 \text{ for all } f \in L^2(\mu)$$

Since $M_{u,T} M^*{}_{u,T} f = u^2 \circ T \cdot f_0 \circ T \cdot E(f)$ and $M^*{}_{u,T} M_{u,T} f = u^2 f_0 f$ we have

$$\int_E \left\{ (u^2 \circ T \cdot f_0 \circ T \cdot E(f))^p - C^2 (u^2 f_0)^p f \right\} d\mu \leq 0 \text{ for every } E \in \Sigma \quad .$$

$\Leftrightarrow\ u^{2p} \circ T\, f_0{}^p \circ T\, E(f) \leq C^2\, u^{2p} f_0{}^p f$ almost everywhere, for all $C \geq 0$

Corollary 4.2

The composition operator C_T on $B(L^2(\mu))$ is p-posinormal if and only if

$f_0{}^p \circ T\, E(f) \leq C^2\, f_0{}^p f$ almost everywhere, for all $C \geq 0$

Proof:

The proof is obtained from Theorem 4.1 by putting $u = 1$.

Theorem 4.3

Let the composite multiplication operator $M_{u,T} \in B(L^2(\mu))$. Then $M_{u,T}$ is k-quasi-posinormal if and only if

$$u f_0\, E(u f_0) \circ T^{-(k-1)}\, (E(u_k)) \circ T^{-k}\, u^2 \circ T^{-(k-1)}\, f_0 \circ T^{-(k-1)} \leq C^2\, (u f_0\, E(u f_0)) \circ T^{-k}\, E(u_{k+1}) \circ T^{-(k+1)}$$

almost everywhere, for all $C \geq 0$.

Proof:

Suppose $M_{u,T}$ is k-quasi-posinormal.

Then $M^{*k}{}_{u,T} (M_{u,T} M^*{}_{u,T}) M^k{}_{u,T} \leq C^2\, M^{*k+1}{}_{u,T} M^{k+1}{}_{u,T}$ for all $C \geq 0$.

This implies that

$$\left\langle (M^{*k}{}_{u,T} (M_{u,T} M^*{}_{u,T}) M^k{}_{u,T} - C^2\, M^{*k+1}{}_{u,T} M^{k+1}{}_{u,T}) f,\ f \right\rangle \leq 0 \text{ for all } f \in L^2(\mu)$$

Since $M_{u,T} M^*{}_{u,T} f = u^2 \circ T \cdot f_0 \circ T \cdot E(f)$

$$\int_E \left\{ \begin{matrix} (u f_0\, E(u f_0)) \circ T^{-(k-1)}\, (E(u_k)) \circ T^{-k}\, u^2 \circ T^{-(k-1)}\, f_0 \circ T^{-(k-1)} - \\ C^2\, (u f_0\, E(u f_0)) \circ T^{-k}\, E(u_{k+1}) \circ T^{-(k+1)}) f \end{matrix} \right\} d\mu \leq 0 \text{ for every } E \in \Sigma \quad .$$

$$\Leftrightarrow \quad u f_0\, E(u f_0) \circ T^{-(k-1)}\, E(E(u_k)) \circ T^{-k}\, u^2 \circ T^{-(k-1)}\, f_0 \circ T^{-(k-1)} \leq C^2\, (u f_0\, E(u f_0)) \circ T^{-k}\, E(u_{k+1}) \circ T^{-(k+1)}$$

almost everywhere, for all $C \geq 0$.

Corollary 4.4

The composition operator C_T on $B(L^2(\mu))$ is k-quasi-posinormal if and only if $f_0\, E(f_0) \circ T^{-(k-1)}\; f_0 \circ T^{-(k-1)} \le C^2\, (f_0\, E(f_0)) \circ T^{-k}$ almost everywhere, for all $C \ge 0$

Proof:

The proof is obtained from Theorem 4.3 by putting $u = 1$.

5. (P, K)-QUASI-POSINORMAL COMPOSITE MULTIPLICATION OPERATORS ON L^2-SPACE

Theorem 5.1

Let the composite multiplication operator $M_{u,T} \in B(L^2(\mu))$. Then $M_{u,T}$ is (p,k)-quasi-posinormal if and only if

$$uf_0\, E(uf_0) \circ T^{-(k-1)}\, (E(u_k)^p) \circ T^{-k}\; u^{2p} \circ T^{-(k-1)}\; f_0{}^p \circ T^{-(k-1)} \le C^2\, uf_0\, E(uf_0)) \circ T^{-(k-1)}\, E(u^{2p} f_0{}^p u_k{}^p) \circ T^{-k}$$

almost everywhere, for all $C \ge 0$.

Proof:

Suppose $M_{u,T}$ is (p,k)-quasi-posinormal.

Then $M^{*k}{}_{u,T}(M_{u,T} M^{*}{}_{u,T})^p\, M^k{}_{u,T} \le C^2\; M^{*k}{}_{u,T}(M^{*}{}_{u,T} M_{u,T})^p\, M^k{}_{u,T}$ for all $C \ge 0$.

This implies that

$$\left\langle \left(M^{*k}{}_{u,T}(M_{u,T} M^{*}{}_{u,T})^p\, M^k{}_{u,T} - C^2\; M^{*k}{}_{u,T}(M^{*}{}_{u,T} M_{u,T})^p\, M^k{}_{u,T}\right) f,\ f \right\rangle \le 0$$

for all $f \in L^2(\mu)$

Since $M_{u,T}\, M^{*}{}_{u,T}\, f = u^2 \circ T \cdot f_0 \circ T \cdot E(f)$

$$\int_E \left\{ \begin{matrix} (uf_0\, E(uf_0) \circ T^{-(k-1)}\, (E(u_k)^p) \circ T^{-k}\; u^{2p} \circ T^{-(k-1)}\; f_0{}^p \circ T^{-(k-1)} \\ -C^2\, uf_0\, E(uf_0)) \circ T^{-(k-1)}\, E(u^{2p} f_0{}^p u_k{}^p) \circ T^{-k})f \end{matrix} \right\} d\mu \le 0$$

for every $E \in \Sigma$.

$\Leftrightarrow$

$$uf_0\, E(uf_0) \circ T^{-(k-1)}\, (E(u_k)^p) \circ T^{-k}\; u^{2p} \circ T^{-(k-1)}\; f_0{}^p \circ T^{-(k-1)} \le C^2\, uf_0\, E(uf_0)) \circ T^{-(k-1)}\, E(u^{2p} f_0{}^p u_k{}^p) \circ T^{-k}$$

almost everywhere, for all $C \ge 0$.

Corollary 4.4

The composition operator C_T on $B(L^2(\mu))$ is (p,k)-quasi-posinormal if and only if $f_0\, E(f_0) \circ T^{-(k-1)}\;\; f_0{}^p \circ T^{-(k-1)} \le\; C^2\, f_0\, E(f_0)) \circ T^{-(k-1)}\, E(f_0{}^p u_k{}^p) \circ T^{-k}$ almost everywhere, for all $C \ge 0$

Proof:

The proof is obtained from Theorem 5.1 by putting $u = 1$.

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