# ON SPECIAL $(\alpha, \beta)$ METRIC WITH A SPECIAL CURVATURE PROPERTIES 

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#### Abstract

In the present paper, we study a class for Finsler Metric called general metric and obtained an equation that characterizes these special Finsler metrics at almost vanishing $\mathrm{H}-$ curvature. As a commencement of this result, we prove that a general special $(\alpha, \beta)$-metric has almost vanishing $H$ Curvature if and only if it is having almost vanishing H-curvature.


Keywords: - Finsler metrics, Berwald space, Douglas space, Douglas space of the second kind, Landsberg space and H -curvature.

## Mathematics Subject Classifications: - 53 B 40

1. INTRODUCTION:- The notion of dually flat metric was first introduced by S. I. Amari and H. Nagaoka, while studying the information geometry on Riemannian spaces [4]. Later, Z. Shen extended the notion of dually flatness to Finsler metrics [13]. Dually flat Finsler metrics form a special important class of Finsler metrics in Finsler information geometry, which play a very important role in studying flat Finsler information structures ([7], [16], [1], [9], and [6]). In 2009, the authors of [5] classified the locally dual flat Randers metrics with almost isotropic flag curvature. Recently, Q. Xia worked on the dual flatness of Finsler metrics of isotropic flag curvature as well as scalar flag curvature ([5], [7]). Also, Q. Xia studied and gave a characterization of locally dually flat ( $\alpha, \beta$ )-metrics on an $n$-dimensional manifold $M(n \geq 3)$ [5]. The Cartan torsion, the S-curvature, the E-curvature and the H-curvature are the examples of few nonRiemannian quantities in Finsler geometry as they vanish for Riemannian metrics. The S-curvature $S(x ; y)$ was introduced by Shen $[6,9]$ and was defined as follows:

$$
S(x ; y)=\frac{d}{d t}\left[\tau\left(\gamma(t), \gamma(t)^{\prime}\right)\right]_{t=0}
$$

where $\tau(x, y)$ is the distortion of the metric $F$ and $\gamma(t)$ is the geodesic with $\gamma(0)=x$ and $\gamma(t)^{\prime}=\mathrm{y}$ on M. In 1988, Akbar-Zadeh introduced H-curvature which is closely related to the S-curvature [1]. The Hcurvature $H_{y}=H_{i j} d x^{i} \otimes d x^{j}$ is defined by

$$
H_{i j}=\frac{1}{4}\left(E_{i j}+E_{j i}\right)
$$

Also F is said to have almost vanishing H -curvature if

$$
H_{i j}=\frac{n+1}{2} \theta F_{y^{i} y^{j}}
$$

Several authors studied the H-curvature of different class of Finsler metrics [10, 12]. In [11], Mo proved that all spherically symmetric Finsler metrics of almost vanishing H-curvature are of almost vanishing _-
curvature and corresponding one forms are exact, generalizing a result previously only known in the case of metrics with vanishing H-curvature. In general, it is difficult to find the Riemann curvature tensor for general $(\alpha, \beta)$-metric. In this paper, we further generalize Mo's result for general $(\alpha, \beta)$-metric under the assumption (1.2).

Theorem 1.1. The general $(\alpha, \beta)$-metric $F=\alpha \phi\left(b^{2} ; s\right)$ satisfying (1.2) has al-most vanishing H-curvature if and only if
$\alpha s\left[(n+1) \frac{\partial R_{1}}{\partial x}+3\left(b^{2}-a^{2}\right)+2(n+1) R_{3}\right]=3(n+1) \theta\left(\phi-s \phi_{s}\right), \theta=\theta_{j}(x) y_{j}$
where $R_{1}, R_{2}$, and $R_{3}$ are given in (2.6), (2.9), and (2.8), respectively. As an application of Theorem 1.1, we have the following corollary.
Corollary 1.2. For the general $(\alpha, \beta)$-metric $F=\alpha \phi\left(b^{2} ; s\right)$ satisfying (1.2) the H-curvature almost vanishes if and only if the $\Xi$-curvature almost vanishes. In this case, the corresponding 1 -form $\theta$ is an exact form.

As a consequence of Corollary 1.2 , for $\theta=0$, we get the following corollary.
Corollary 1.3. For the general ( $\alpha, \beta$ )-metric $F=\alpha \phi\left(b^{2} ; s\right)$ satisfying (1.2) the H-curvature vanishes if and only if the $\Xi$-curvature vanishes.

A Finsler metric is said to be R-quadratic if its Riemann curvature $R_{y}$ is quadratic in $y \in T_{m} X$. These Rquadratic Finsler metrics always have vanishing H-curvature [10]. Together with Corollary 1.3, we have the following.
Corollary 1.4. The $\Xi$-curvature of a R-quadratic general $(\alpha, \beta)$-metric always vanishes.
Let M be an n -dimensional smooth manifold $T_{x} M$ denotes the tangent space of M at x . The tangent bundle M is the union of tangent spaces $T_{x} M:=\mathrm{U}_{x \in M} T_{x} M$. We denote the elements of TM by ( $\mathrm{x}, \mathrm{y}$ ) where $y \in T_{x} M$ and $T M_{0}:=T M \backslash\{0\}$.

In this paper, we study and characterize curvature of special $(\alpha, \beta)$ Finsler metric $F=\alpha+\beta+\frac{\beta^{2}}{\alpha}+$ $\frac{\beta^{3}}{\alpha^{2}}$ with isotropic S-curvature, which is not Riemannian.

## 2. Preliminaries:-

Let $M$ be an n-dimensional smooth manifold. We denote by TM the tangent bundle of M and by ( $\mathrm{x}, \mathrm{y}$ ) = $\left(x^{i}, y^{j}\right)$ the local coordinates on the tangent bundle TM. A Finsler manifold ( $M, F$ ) is a smooth manifold equipped with a function $\mathrm{F}: \operatorname{TM} \rightarrow[0, \infty)$, which has the following properties:

- Regularity: $F$ is smooth in $T M \backslash\{0\}$;
- Positive homogeneity: $F(x, \lambda y)=\lambda F(x, y), \forall \lambda>0$,
- Strong convexity: the Hessian matrix of $F^{2}, g_{i j}=\frac{1}{2}\left(\frac{\partial^{2} F^{2}(x, y)}{\partial x^{i} \partial y^{j}}\right)$ is positive definite on $\mathrm{TM} \backslash\{0\}$. We call F and the tensor $g_{i j}$ the Finsler metric and the fundamental tensor of M , respectively.

For a Finsler metric $F=F(x, y)$, its geodesic curves are characterized by the system of differential equations $\ddot{c}^{i}+2 G^{i}(\dot{c})=0$, where the local functions $G^{i}=G^{i}(\mathrm{x}, \mathrm{y})$ are called the spray coefficients and given by

$$
G^{i}=\frac{g^{i l}}{4}\left\{\left[F^{2}\right]_{x^{k} y^{l}} y^{k}-\left[F^{2}\right]_{x} l\right\}, \forall \mathrm{y} \in T_{x} \mathrm{M}
$$

Definition 2.1:- A Finsler metric $F=F(x, y)$ on a manifold M is said to be locally dually flat if at any point there is a standard coordinate system $\left(x^{i}, y^{i}\right)$ in TM which satisfies

$$
\left(F^{2}\right)_{x^{k} y l} y l=2\left(F^{2}\right)_{x^{l}}
$$

In this case, the system of coordinates $\left(x^{i}\right)$ is called an adapted local coordinate system. It is easy to see that every locally Minkowskian metric is locally dually flat. But the converse is not generally true [7].
Definition 2.2:- A Finsler metric is said to be locally projectively flat if at any point there is a local coordinate system in which the geodesics are straight lines as point sets. It is known that a Finsler metric $\mathrm{F}(\mathrm{x}, \mathrm{y})$ on an open domain $\mathrm{U} \subset R^{n}$ is locally projectively flat if and only if its geodesic coefficients $G^{i}$ are of the form

$$
G^{i}=P y^{i}
$$

where $P: T U=U \times R^{n} \rightarrow R$ is positively homogeneous of degree one, $P(x, y)=\lambda P(x, y), \forall \lambda>0$. We call $\mathrm{P}(\mathrm{x}, \mathrm{y})$ the projective factor of $\mathrm{F}(\mathrm{x}, \mathrm{y})$.
Lemma 2.1:- ([7]). Let $F=F(x, y)$ be a Finsler metric on an open subset $\mathrm{U} \subset R^{n}$. Then F is locally flat and projectively flat on U if and only if $F_{x^{i}}=\mathrm{CF} F_{y^{l}}$, where C is a constant. The S-curvature is a scalar function on TM, which was introduced by Z. Shen to study volume comparison in Riemann-Finsler geometry [10]. The S-curvature measures the average rate of change of $\left(T_{x} M F_{x}=F \mid T_{x} M\right)$ in the direction y $\in T_{x} M$. It is known that $\mathrm{S}=0$ for Berwald metrics.
Definition2.3. A Finsler metric $F$ on an $n$-dimensional manifold $M$ is said to have isotropic $S$-curvature if $S$ $=(\mathrm{n}+1) \mathrm{c}(\mathrm{x}) \mathrm{F}$, for some scalar function c on M .
For a Finsler metric F on an n-dimensional manifold M, the Busemann-Hausdorff volume form
$d V_{F}=\sigma_{F(x)} d x^{i} \ldots \ldots . d x^{n}$ is defined by

$$
\sigma_{F}=\frac{\operatorname{Vol}\left(B^{m}(1)\right)}{\operatorname{Vol}\left\{\left(y^{i}\right) \in R^{n} \left\lvert\, F\left(x,\left.y^{i} \frac{\partial}{\partial x^{i}}\right|_{x}<1\right\}\right.\right.}
$$

Here Vol denotes the Euclidean volumes and $B^{n}(1)$ denotes the unit ball in $R^{n}$. Then the S-curvature is defined by

$$
S=\frac{\partial G^{i}}{\partial y^{i}}(x, y)-y^{i} \frac{\partial\left(\operatorname{In} \sigma_{F}\right)}{\partial x}
$$

where

$$
y=\left.y^{i} \frac{\partial\left(\operatorname{In} \sigma_{F}\right)}{\partial x^{i}}\right|_{x} \in T_{x} M \text { [7]. }
$$

For an $(\alpha, \beta)$-metric, one can write $F=\alpha \phi(s)$, where $s=\beta / \alpha$ and $\phi=\phi(s)$ is a $C^{\infty}$ function on the interval $\left(-b_{0}, b_{0}\right)$ with certain regularity properties, $\alpha=\sqrt{a_{i j} y^{i} y^{j}} \quad$ is a Riemannian metric and $\beta=\beta=$ $b_{i(x)} y^{i}$ is a 1 -form on $M$.
We further denote

$$
b_{i \mid j} \theta^{j}=d b_{i}-b_{j} \theta_{i}^{j}
$$

where $\theta^{i}=d x^{i}$ and $\theta_{j}^{i}=\Gamma_{i k}^{j} d x^{k}$ denotes the coefficients of the Levi- Civita connection form of $\alpha$. Let

$$
r_{i j}=\frac{1}{2}\left(b_{i \mid j}+b_{j \mid i}\right), \quad s_{i j}=\frac{1}{2}\left(b_{i \mid j}-b_{j \mid i}\right)
$$

Clearly, $\beta$ is closed if and only if $s_{i j}=0$. An $(\alpha, \beta)$-metric is said to be trivial if $b_{i j}=s_{i j}=0$. we put

$$
\begin{array}{ll}
r_{i 0}=r_{i j} y^{j}, & r_{00}=r_{i j} y^{i} y^{j} \quad, \quad r_{j}=r_{i j} b^{i} \\
s_{i 0}=r_{i j} y^{j}, & s_{j}=s_{i j} b^{j}, r_{0}=r_{j} y^{j}, s_{0}=s_{j} y^{j}
\end{array}
$$

By direct computation, we can obtain a formula for the mean Cartan torsion of an $(\alpha, \beta)$-metric as follow:

$$
I_{i}=-\frac{\Phi\left(\phi-s \phi^{\prime}\right)}{2 \Delta \phi \alpha^{2}}\left(\alpha b_{i}-s y_{i}\right)
$$

Clearly, an $(\alpha, \beta)$-metric $F=\alpha \phi(s), s=\beta / \alpha$ is Riemannian if and only if $\Phi=0$. Hence, we further we assume that $\Phi \neq 0$.
Theorem 2.2. [98] Let $F=\alpha \phi(s), s=\beta / \alpha$ be a $(\alpha, \beta)$-metric on an n-dimensional manifold $M_{n}(n \geq$ 3), where $\alpha=\sqrt{a_{i j} y^{i} y^{j}}$ is a Riemannian metric and $\beta=b_{i(x)} y^{i} \neq 0$ is a 1 -form on M. Suppose that F is not Riemannian and $\phi^{\prime}(s) \neq 0$. Then F is locally dually flat on M if and only if $\alpha, \beta$ and $\phi=\phi(s)$ satisfy

1. $s_{l 0}=\frac{1}{3}\left(\beta \theta_{l}-\theta b_{l}\right)$,
2. $r_{00}=\frac{2}{3} \theta \beta+\left[\tau+\frac{2}{3} b^{2} \tau-\theta_{l} b^{l}\right] \alpha^{2}+\frac{1}{3}\left(3 k_{2}-2-3 k_{3} b^{2}\right) \tau \beta^{2}$,
3. $G_{\alpha}^{l}=\frac{1}{3}\left[2 \theta+\left(3 k_{1}-2\right) \tau \beta\right] y^{l}+\frac{1}{3}\left(\theta^{l} \tau b^{l}\right) \alpha^{2}+\frac{1}{2} k_{3} \tau \beta^{2} b^{l}$,
4. $\tau\left[s\left(k_{2}-k_{3} s^{2}\right)\left(\phi \phi^{\prime}-s \phi^{2}-s \phi \phi^{\prime \prime}\right)-\left(\phi^{2}+\phi \phi^{\prime \prime}\right)+k_{1} \phi\left(\phi-s \phi^{\prime}\right)\right]=0$.
where $\tau=\tau(x)$ is a scalar function, $\theta=\theta_{i}(x) y^{i}$ is an 1-form on $M, \theta^{l}=\alpha^{l m} \theta_{m}$,

$$
\begin{gathered}
k_{1}=\Pi(0), \quad k_{2}=\frac{\Pi^{\prime}(0)}{Q(0)} \quad, \quad k_{3}=\frac{1}{6 Q^{2}(0)}\left[3 Q^{\prime \prime}(0) \Pi^{\prime}(0)-6 \Pi(0) 2-Q(0) \Pi^{\prime \prime \prime}(0)\right] \\
Q
\end{gathered}=\frac{\phi^{\prime}}{\phi-s \phi^{\prime}} \quad, \quad \Pi=\frac{\phi \prime 2+\phi \phi^{\prime \prime}}{\phi\left(\phi-s \phi^{\prime}\right)} .
$$

and
In [22], Cheng-Shen studied the class of $(\alpha, \beta)$-metrics of non-Randers type $\phi \neq t_{1} \sqrt{1+t_{2} s^{2}}+$ $t_{3} s$ with isotropic S-curvature and obtained the following
Theorem 2.3:- ([10]). Let $F=\alpha \phi(s), s=\beta / \alpha$ be a non-Riemannian $(\alpha, \beta)$-metric on a manifold and $b=\left\|\beta_{x}\right\| \alpha$. suppose that $\phi \neq t_{1} \sqrt{1+t_{2} s^{2}}+t_{3} s$ for any constants $t_{1}>0, t_{2}$ and $t_{3}$. Then F is of isotropic S-curvature $S=(n+1) c F$ if and only if one of the following assertions holds
i) $\beta$ Satisfies

$$
\begin{equation*}
r_{i j}=\varepsilon\left\{b^{2} a_{i j}-b_{i} b_{j}\right\}, s_{j}=0, \tag{5.2.1}
\end{equation*}
$$

where $\varepsilon=\varepsilon(x)$ is a scalar function, and $\mathrm{c}=\mathrm{c}(\mathrm{x})$ satisfies

$$
\begin{equation*}
\Phi=-2(n+1) k \frac{\phi \Delta^{2}}{b^{2}-s^{2}} \tag{5.2.2}
\end{equation*}
$$

where k is a real constant. In this case, $S=(n+1) c F$ with $c=k \varepsilon$.
ii) $\beta$ Satisfies
(5.2.3) $\quad r_{i j}=0, s_{i j}=0$.

In this case, $S=0$, regardless of the choice of a particular $\phi$.

## 3. Characterization of locally dually flat first approximate Special metric

Theorem 3.1. Let $F=\alpha+\beta+\frac{\beta^{2}}{\alpha}+\frac{\beta^{3}}{\alpha^{2}}$ be a first approximate Matsumoto metric on a manifold M of dimension $n \geq 3$.Then the necessary and sufficiency conditions for $F$ to be locally dually flat on $M$ are the following:
5. $s_{l 0}=\frac{1}{3}\left(\beta \theta_{l}-\theta b_{l}\right)$,
6. $r_{00}=\frac{2}{3} \theta \beta+\left[\tau+\frac{2}{3}\left(b^{2} \tau-\theta_{l} b^{l}\right)\right] \alpha^{2}+\frac{1}{3}\left(25-30 b^{2}\right) \tau \beta^{2}$,
7. $G_{\alpha}^{l}=\frac{1}{3}[2 \theta+25 \tau \beta] y^{l}+\frac{1}{3}\left(\theta^{l} \tau b^{l}\right) \alpha^{2}+\frac{10}{2} \tau \beta^{2} b^{l}$,
where $\tau=\tau(x)$ is a scalar function, $\theta=\theta_{k} y^{k}$ is an 1-form on $M$.
Proof: - For a Finsler metric $F=\alpha+\beta+\frac{\beta^{2}}{\alpha}+\frac{\beta^{3}}{\alpha^{2}}$ we obtain $k_{1}=3, k_{2}=9, k_{3}=10$, and

$$
\begin{gathered}
\phi=1+s+s^{2}+s^{3}, \quad \phi^{\prime}=1+2 s+3 s^{2}, \quad \phi^{\prime \prime}=2+6 s, \quad \phi^{\prime}=6 \\
\Pi=\frac{3+12 s+18 s^{2}+20 s^{3}+13 s^{4}}{1+s-2 s^{3}-3 s^{4}-3 s^{5}-2 s^{6}}, \Pi(0)=3, \Pi^{\prime}(0)=9, \Pi^{\prime \prime}(0)=18, \\
\Pi^{\prime \prime \prime}(0)=102 \\
Q=\frac{1+2 s+3 s^{2}}{1-s^{2}-2 s^{3}}, \quad Q^{\prime}=\frac{2+8 s+8 s^{2}+8 s^{3}+6 s^{4}}{\left(-1+s^{2}+2 s^{3}\right)^{2}}, \\
Q^{\prime \prime} \frac{-8\left(1+8 s+9 s^{2}+15 s^{3}+9 s^{4}+6 s^{5}+3 s^{6}\right)}{\left(-1+s^{2}+2 s^{3}\right)^{3}} \\
Q(0)=1, Q^{\prime}(0)=2, Q^{\prime \prime}(0)=8, Q^{\prime \prime \prime}(0)=24
\end{gathered}
$$

By using the above values in Lemma 2.1, we get

$$
\left[s\left(k_{2}-k_{3} s^{2}\right)\left(\phi \phi^{\prime}-s \phi^{\prime 2}-s \phi \phi^{\prime \prime}\right)-\left(\phi^{\prime} 2+\phi \phi^{\prime \prime}\right)+k_{1} \phi\left(\phi-s \phi^{\prime}\right)\right]=0, \text { and } \tau=0
$$

Then, finally, by substituting $k_{1}, k_{2}$ and $k_{3}$ in Lemma 2.1, we infer the claim
Now, let $\phi=\phi(s)$ be a positive $C^{\infty}$ function on $\left(-b_{0}, b_{0}\right)$. For a number $b \in\left[0, b_{0}\right]$,
Let

$$
\begin{equation*}
\Phi=-\left(Q-s Q^{\prime}\right) \cdot(n \Delta+1+s Q)-\left(b^{2}-s^{2}\right)(1+s Q) Q^{\prime \prime} \tag{5.3.1}
\end{equation*}
$$

where $\quad \Delta=1+s Q+\left(b^{2}-s^{2}\right) Q^{\prime}$. This implies that

$$
\Delta=\frac{\phi\left\{1-3 s^{2}-8 s^{3}+2 b^{2}(6+2 s)\right\}}{\left(-1+s^{2}+2 s^{3}\right)^{2}}
$$

Then the equation (5.3.1) can be written as follows:

$$
\Phi=-\left(Q-s Q^{\prime}\right)(n+1) \Delta+\left(b^{2}-s^{2}\right)\left\{\left(Q-s Q^{\prime}\right) Q^{\prime}-(1+s Q) Q^{\prime \prime}\right\}
$$

By using Theorem 2.3 , now we will consider a locally dually flat $(\alpha, \beta)-$ metric with isotropic S -curvature.
Theorem3.2. Let $F=\alpha+\beta+\frac{\beta^{2}}{\alpha}+\frac{\beta^{3}}{\alpha^{2}}$ be a locally dually flat non-Randers type $(\alpha, \beta)$-metric on a manifold M of dimension $n \geq 3$. Suppose that F is of isotropic S-curvature $S=(n+1) c F$, where $c=$ $c(x)$ a scalar function is on M . Then F is a locally projectively flat in adapted coordinate system and $G^{i}=0$. Proof. Let $G^{i}=G^{i}(x, y)$ and $\underline{G}_{\alpha}^{i}=\underline{G}_{\alpha}^{i}(x, y)$ denote the coefficients of F and $\alpha$ respectively, in the same coordinate system. By definition, we have

$$
\begin{equation*}
G^{i}=G_{\alpha}^{i}+P y i+Q^{i} \tag{5.3.2}
\end{equation*}
$$

where

$$
\begin{equation*}
P=\alpha^{-1} \theta-2 Q \alpha s_{0}+r_{00} \tag{5.3.3}
\end{equation*}
$$

$$
\begin{gather*}
Q^{i}=\alpha Q s_{0}^{i}+\Psi-2 Q \alpha s_{0}+r_{00} b^{i},  \tag{5.3.4}\\
\Theta=\frac{\phi \phi^{\prime}-s\left(\phi \phi^{\prime \prime}+\phi^{\prime} \phi^{\prime}\right)}{2 \phi\left(\left(\phi-s \phi^{\prime}\right)+\left(b^{2}-s^{2}\right) \phi^{\prime \prime}\right)}=\frac{-1+s+12 s^{2}+20 s^{3}+21 s^{4}+3 s^{5}}{2 \phi\left\{-1+2 s^{2}+3 s^{3}-2 b^{2}(1+2 s)\right\}} \\
\Psi=\frac{1}{2} \frac{\phi^{\prime \prime}}{\left(\phi-s \phi^{\prime}\right)+\left(b^{2}-s^{2}\right) \phi^{\prime \prime}}=\frac{1+3 s}{1-3 s^{2}-8 s^{3}+b^{2}(2+6 s)}
\end{gather*}
$$

First, we suppose that case (i) of Theorem 2.3 holds. It is remarkable that, for a
1st approximation Matsumoto metric, we have

$$
\Delta=\frac{\phi\left\{1-3 s^{2}-8 s^{3}+2 b^{2}(6+2 s)\right\}}{\left(-1+s^{2}+2 s^{3}\right)^{2}}
$$

It follows that $\left(-1+s^{2}+2 s^{3}\right)^{2} \Delta$ is a polynomial in s of degree 3 . On the other hand we have

$$
\begin{equation*}
\phi \Delta^{2}=\frac{\phi^{2}\left\{1-3 s^{2}-8 s^{3}+2 b^{2}(6+2 s)\right\}^{2}}{\left(-1+s^{2}+2 s^{3}\right)^{4}} \tag{5.3.5}
\end{equation*}
$$

Hence, if case (ii) of Theorem (5.2.3) holds, then substituting (5.3.5) we obtain that
(5.3.6) $\left(b^{2}-s^{2}\right)\left(-1+s^{2}+2 s^{3}\right)^{4} \Phi=-2(n+1) k \phi^{2}\left\{1-3 s^{2}-8 s^{3}+2 b^{2}(6+2 s)\right\}^{2}$.

It follows that $\left(b^{2}-s^{2}\right)\left(-1+s^{2}+2 s^{3}\right)^{4} \Phi$ is not a polynomial in s (if $\mathrm{k}=0$, then by considering the Cartan torsion equation, we get a contradiction). Then, we put

$$
\begin{aligned}
& \phi \Delta^{2}=\frac{\underline{\Delta}}{\left(-1+s^{2}+2 s^{3}\right)^{4}} \\
& \text { where } \\
& \underline{\Delta}=\phi^{2}\left\{1-3 s^{2}-8 s^{3}+2 b^{2}(6+2 s)\right\}^{2}
\end{aligned}
$$

By assumption, F is a non-Randers type metric. Thus $\underline{\Delta}$ is not a polynomial in s, and then $\left(b^{2}-s^{2}\right)(-1+$ $\left.s^{2}+2 s^{3}\right)^{4} \Phi$ is not a polynomial in s. Now, let us consider another form of $\Phi$ :

$$
\Phi=-\left(Q-s Q^{\prime}\right)(n+1) \Delta+\left(b^{2}-s^{2}\right)\left\{\left(Q-s Q^{\prime}\right) Q^{\prime}-(1+s Q) Q^{\prime \prime}\right\}
$$

where

$$
Q-s Q^{\prime}=\frac{1-6 s^{2}-12 s^{3}-15 s^{4}-12 s^{5}}{\left(-1+s^{2}+2 s^{3}\right)^{2}}
$$

Then
(5.3.7)

$$
\begin{aligned}
& \Phi=\frac{-1}{\left(-1+s^{2}+2 s^{3}\right)^{4}}\left[\phi \left[1-15 s^{2}-38 s^{3}-81 s^{4}-108 s^{5}-33 s^{6}-6 s^{7}\right.\right. \\
& +\quad n\left(-1-9 s^{2}-20 s^{3}+3 s^{4}+72 s^{5}+141 s^{6}+156 s^{7}+96 s^{8}\right)+2 b^{2}\left\{4 \left(1+3 s+9 s^{2}\right.\right. \\
& \left.\left.+15 s^{3}+9 s^{4}+6 s^{5}+3 s^{6}-n\left(-1-3 s+6 s^{2}+30 s^{3}+51 s^{4}+57 s^{5}+36 s^{6}\right)\right\}\right]
\end{aligned}
$$

From equations (5.3.6) and (5.3.7), the relation $\left(b^{2}-s^{2}\right)\left(-1+s^{2}+2 s^{3}\right)^{4} \Phi$ is a polynomial in $s$ and $b$ of degree 8 and 4 respectively. The coefficient of $s^{8}$ is not equal to zero. Hence it's impossible that $\Phi=0$. Therefore, we can conclude that equation (5.2.2) does not hold. So, the case (ii) of Theorem 2.3 holds. In this case, we have

$$
r_{00}=0, \quad s_{j}=0
$$

In Theorem 3.1(2), by taking $r_{00}=0$, we obtain

$$
\begin{equation*}
\left[\tau+\frac{2}{3}\left(b^{2} r-\theta_{l} b^{l}\right)\right] \alpha^{2}=\frac{1}{3} \beta\left[-2 \theta-\left(25+30 b^{2}\right) \beta r\right] \tag{5.3.8}
\end{equation*}
$$

Since $\alpha^{2}$ is an irreducible polynomial of $y^{i}$, equation (5.3.8) reduces to the following

$$
\begin{equation*}
\tau+\frac{2}{3}\left(b^{2} r-b_{m} \theta^{m}\right)=0 \tag{5.3.9}
\end{equation*}
$$

$$
\begin{equation*}
\frac{2}{3} \theta+\frac{1}{3}\left(25+30 b^{2}\right) \beta \tau=0 \tag{5.3.10}
\end{equation*}
$$

whence

$$
\begin{equation*}
\theta=-\frac{1}{2}\left(25+30 b^{2}\right) \beta \tau \tag{5.3.11}
\end{equation*}
$$

Then Theorem 3.1(1) yields

$$
s_{0}=-\frac{1}{3\left(b^{2} r-\beta b_{m} \theta^{m}\right)}
$$

This implies

$$
b^{2} r-\beta b_{m} \theta^{m}=0
$$

From (5.3.8), (5.3.9) and (5.3.11), we obtain

$$
\begin{equation*}
\theta=-\frac{1}{2}\left(25+30 b^{2}\right) \beta \tau \tag{5.3.12}
\end{equation*}
$$

From equations (5.3.9) and (5.3.12), it follows that $\tau=0$ and substituting $\tau=0$ in equation (5.3.12), we get $\theta$ $=0$. Thus finally (1), (2) and (3) reduce to the following

$$
s_{i j}=0, G_{\alpha}^{l}=0, r_{00}=0
$$

Since $s_{0}=r_{00}=0$, then equations (5.3.3) and (5.3.4) reduce to

$$
P=0 \text { and } Q^{i}=0 .
$$

Then the relation (5.3.2) becomes $G_{\alpha}^{l}=0$, which completes the proof.
Theorem:-3.3. Let $F=\alpha+\beta+\frac{\beta^{2}}{\alpha}+\frac{\beta^{3}}{\alpha^{2}}$ be a non-Riemannian metric on n-dimensional ( $n \geq 3$ ) manifold M. Then F is locally dually flat with isotropic S-curvature. Moreover, $S=(n+1) c F$ if and only if the structure is locally Minkowskian. Proof. From Theorem 3.2 we have that $F=\alpha+\beta+\frac{\beta^{2}}{\alpha}+\frac{\beta^{3}}{\alpha^{2}}$ is dually flat and projectively flat in any adapted coordinate system. By Lemma 2.1, we infer

$$
F_{x^{k}}=C F F_{y^{k}} .
$$

Hence the spray coefficients $G^{i}=P y^{i}$ are given by $P=\frac{1}{2} C F$. since $G^{i}=0$, then $\mathrm{P}=0$, and hence $\mathrm{C}=$ 0 . This implies that $F_{x^{k}}=0$ and then F is a locally Minkowskian metric in the adapted coordinate system.

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