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The Mathematical Formulas for Creation Annihilation Operators

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Abstract
The aim of the study is to use the creation and annihilation operators in understanding the mathematical structure of the quantum field theory and the process of quantizing the quantum field. It also aims to find mathematical formulas for the wave function space when these operators are repeated. with a particular focus on their application to the Dirac delta function relations and the quantization of a scalar field. The study involves the derivation and analysis of differential equations representing the evolution of quantum states, leading to the establishment of mathematical formulas for creation and annihilation operators. The significance of these operators is underscored by their integral role in describing the quantum nature of scalar fields. Results, to construct the quantization process in the quantum field using the creation and annihilation operations in a simplified mathematical way in order to simplify the mathematical understanding of the of the quantum field theory. The study also obtained to find a mathematical formula for the future of the wave function space when the effects of creation and annihilation are repeated.
Keywords: Mathematical Formulas, Creation Annihilation, quantum states, Operators, Dirac delta function.

1. Introduction

Creation and annihilation operators are mathematical entities widely utilized in quantum mechanics, particularly in examining quantum harmonic oscillators and many-particle systems. An annihilation operator, typically represented as 'a,' reduces the particle count in a specific state by one [1]. On the other hand, a creation operator, (usually denoted a^+) raises the particle count in a given state by one, serving as the adjoint of the annihilation operator [2]. In certain branches of physics and chemistry, the substitution of these operators for wavefunctions is termed second quantization [3, 4].

Creation and annihilation operators are applicable to states of diverse particle types. In quantum chemistry and many-body theory, these operators are frequently applied to electron states. Additionally, they can be specifically associated with the ladder operators for the quantum harmonic oscillator [5, 6]. In this context, the raising operator is construed as a creation operator, introducing a quantum of energy to the oscillator system (similarly for the lowering operator) [7]. These operators can also be employed to depict phonons [8]. The mathematical formulation for bosonic creation and annihilation operators mirrors that of the ladder operators in the quantum harmonic oscillator [9]. Specifically, the commutator of the creation and annihilation operators linked to the same bosonic state equals one, while all other commutators become zero

[10]. Conversely, for fermions, the mathematical treatment is distinct, involving anti-commutators instead of commutators [11, 12].

2. Dirac delta function (relations)

We will present some relations of Dirac, which is called the Dirac delta function. We will use in the quantization of quantum field process later [13,14].

1) $\delta_a \ (x) = \text{Lim} \ \frac{1}{a\sqrt{\pi}} e^{\frac{-x^2}{2}}$

2) $\delta \ (x - a \ ) = \ \frac{1}{2\pi} \ \ \int e^{ip(x-a\ )} \ \ d_p$

3) $\delta \ (x + a \ ) = \ \ \int_{\infty}^{\infty} F(x)\delta \ (x - a \ ) = F \ (a)$

4) $\delta \ (\zeta - \eta \ ) = \ \int_{\infty}^{\infty} \delta \ (\zeta - \eta)\delta \ (x - \eta) \ dx$

5) $\delta \ (\alpha \ x) = \ |\alpha \ |^{-n} \delta \ (\ x)$

$\delta^{\backslash} (-x) \ = \ - \delta^{\backslash} \ (x)$

$x \ \delta^{\backslash} \ (x) = \ - \delta^{\backslash} \ (x)$

3. Quantization of scalar field

Consider the particle (boson has spin = 0) [15, 16]. In historical in quantum mechanics

$$[q_j \ p_j] = \ \delta_{ij} \tag{1}$$

$$[p_{j\ ,}p_j] \ = \ 0 \ , [q_{i\ ,}q_j] \ = \ 0 \tag{2}$$

Now we transfer this idea to use in constructing the quantum field (Q . F).
$q_j \ \rightarrow \ \varphi \ (x) \quad p_j \ \rightarrow \ \pi \ (x)$

Such that

$$[\varphi(x) \ , \pi \ (x) \ ] = i, \delta^3 \ (x - \ x^{\backslash}) \tag{3}$$

where $\delta^3 \ (x - x^{\backslash})$ is Dirac delta function

When performing the quantization process must be calculation spectrum field. By generating the state using oscillator harmonic:
Start for Fourier transform

$$\emptyset \ (x, t) = \ \int \frac{d^3 p}{(2\pi)^3} e^{ip_i x} \ \emptyset \ (p, t) \tag{4}$$

And from solar field equation

$$[H_{SHO} \ \ , a^+ \ ] = \ - \ \omega \ a^+ \tag{5}$$

$$| \ n\rangle = \ (a^+)^n |o\rangle \tag{6}$$

To build the quantization process for the quantum field, we now use the same method, namely, the creation and annihilation operators.
Use this is scalar field equation.
We how that $(\boxdot^2 + \ m^2) \ \emptyset \ (t, \ x) = o$

And it is not harmonic oscillator, must make fore this clearly harmonic oscillator and – spectrum [2].

$$(\boxdot^2 + \ m^2) \ \emptyset \ (t, x) = \int \frac{d^3}{(2\pi)^3} \ [(\boxdot^2 + \ m^2) \ \emptyset \ (t, x)[e^{ip_i x} \emptyset \ (p, t)]$$

$$= \int \frac{d^3p}{(2\pi)^3} \left[\frac{d^2}{dx^2} \ |p|^2 + m^2\right] \emptyset(p,t) = 0$$

$$= \left[\frac{d^2}{dx^2} + [|p|^2] + m^2\right] \emptyset(p,t) = 0 \tag{7}$$

This is harmonic oscillator fore Klein Gordon equation

$$\omega_p = \sqrt{p^n + m^n}$$

Now we can write Harmonic oscillator for structure of Quantum Field [3].

We use $\frac{d^3p}{(2\pi)^3} \times \frac{1}{\sqrt{2\omega p}}$ lorentes inverient

Element in group field theory (GFT) scalar field [4].

$$\emptyset(x) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2\omega_p}} \left[a_p e^{ip_i x} + a^+ e^{-ip_i x}\right] \frac{d^3p}{(2\pi)^3} \tag{8}$$

$$\pi[x] = \int \frac{d^3p}{(2\pi)^3} (-1) \int \sqrt{\frac{\omega_p}{2}} \left[a_p e^{ip_i x} - a^+ e^{-ip_i x}\right] \tag{9}$$

And use

$$\left(\frac{d^2}{dt^2} + p^2 + m^2\right) \emptyset(p,t) = 0 \tag{10}$$

Using $[\emptyset(x), \pi(x)] = i\delta^3 (x - x^\backslash)$

$$\Rightarrow [\emptyset(x), \pi(x)] = \int \frac{d^3p\, d^3_{p^\backslash}}{(z\pi)^6} \times - \frac{i}{2} \int \frac{\omega^\backslash}{wp} - \left[a_p, a^+_{-p^\backslash}\right] e^{i[px + p^\backslash x^\backslash]}$$
$$= i\delta^3 (x - x^1) \tag{11}$$

Using the equations (10) , (11) and 2,3 from Dirac delta functions

$$\left[a_p, a^+_{-p^\backslash}\right] = 2\pi^2 \ \delta^3 (p - p^\backslash)$$

Commutation relation for creation and scalar field

Using equation (11)

$$\mathcal{H} = \int d^3 x\, \mathcal{H} = \int d^3 x \left[-\tfrac{1}{2}\pi^2 + \tfrac{1}{2}(m\emptyset)^2 + \tfrac{1}{2}(\nabla\emptyset)^2\right. \tag{12}$$

$$\mathcal{H} = \int d^3 x \int \frac{d^3p\, d^3_{p^\backslash}}{(z\pi)^6} \times e^{i[p+p^\backslash]x} \left[-\frac{\sqrt{\omega_p \omega_p^\backslash}}{4}\right] \tag{13}$$

$$\left[a_p, -a^+_{-p^\backslash}\right]\left[a_{p^\backslash}, -a^+_{p^\backslash}\right] + \frac{-pp^\backslash + m^2}{4\sqrt{\omega_p \omega_p^\backslash}} \left[a_p, +a^+_{-p^\backslash}\right]\left[a_{-p^\backslash}, +a^+_{p^\backslash}\right] \tag{14}$$

$$= (2\pi)^3 \int \frac{d^3p\, d^3_{p^\backslash}}{(z\pi)^6} \times \delta(p + p^\backslash)\left[-\frac{\sqrt{\omega_p \omega_p^\backslash}}{-1}\left[a_p, -a^+_p\right]\left[a_p, +a^+_{-p^\backslash}\right] \frac{-pp^\backslash + m^2}{\sqrt{\omega_p \omega_p^\backslash}} \left[a_p, -a^+_{-p}\right]\left[a_p, -a^+_{p^\backslash}\right] =\right.$$

$$\int \frac{d^3p\, d^3_{p^\backslash}}{(z\pi)^6} \frac{-\omega_p}{4} \left[a_p, -a^+_{-p}\right]\left[a_{-p}, -a^+_{-p^\backslash}\right] \tag{15}$$

$$\mathcal{H} = \int \frac{d^3p}{(z\pi)^3}\ \omega_p[a_p a_p^+ + \frac{1}{2}[a_p, \quad a_p^+] \tag{16}$$

Such that $\frac{1}{2}[a_p, \ a_p^+]$ is vacuum state

$$[a_p, \ a_p^+] = I\delta^3(0) \tag{17}$$

$[\mathcal{H}, a^+] = \omega_p a_p^+$ creation of particles [quantize]

$$[\mathcal{H}, a] = -\omega_p a_p \tag{18}$$

In equations (17) and (18) we obtained to quantize of quantum field

Proposition (1):

If φ is an eigenvector for $a\ a^+$ with eigenvalue λ, then...

$$a^+a(a\varphi) = (\lambda - 1)a\varphi \tag{19}$$

$$a^+a\ (a^+\varphi) = (\lambda + 1)a^+\varphi \tag{20}$$

Hence, either $a\varphi$ is zero, or $a\varphi$ is an eigenvector for a^+a with an eigenvalue $\lambda - 1$. Similarly, either $a^+\varphi$ is zero, or $a^+\varphi$ is an eigenvector for $a^+\varphi$ with an eigenvalue $\lambda - 1$. In other words, the operators a^* and a raise and lower the eigenvalues of a^*a, respectively [10].

Proof

Using the commutation relation (19) we find that

$$a^+a(a\varphi) = a\ (a^+a) - a)\varphi = (\lambda - 1)a\varphi$$

A similar calculation applies to $a^+\varphi$, using (2.30) .

If φ is an eigenvector for a^+a with eigenvalue λ, then

$$\lambda\langle\varphi, \varphi\rangle = \langle\varphi, a^+a\varphi\rangle = \langle a\varphi, a\varphi\rangle \geq 0$$

which means that λ ≥ 0. Let us assume that a^+a has at least one eigenvectorφ, with eigenvalue λ, which we expect since a^*a is self-adjoint. Since a lowers the eigenvalue of a^+a, if we apply a repeatedly to φ, we must eventually get zero. After all, if an φ were always nonzero, these vectors would be, for large n, eigenvectors for $a^*a$with negative eigenvalue, which we have seen is impossible.
It follows that there exists some N ≥ 0 such that $a^N\varphi \neq 0$ but $a^{N+1}\varphi$ =0. If we define $\varphi_\circ$ by

$$\varphi_\circ = a^N \tag{21}$$

then a$\varphi_\circ$ = 0, which means that $a^*a\varphi_\circ$ = 0. Thus, $\varphi_\circ$ is an eigenvector for a^*a with eigenvalue 0. (It follows that the original eigenvalue λ must have been equal to the non-negative integer N.) The conclusion is this: Provided that a^*a has at least one eigenvector φ, we can find a nonzero vector $\varphi_\circ$ such that

$$a\varphi_\circ = a^+a\varphi_\circ = 0$$

Since a^*a cannot have negative eigenvalues, we may call $\varphi_\circ$a "ground state" for a^*a, that is, an eigenvector with lowest possible eigenvalue.We may then apply the raising operator a^* repeatedly to $\varphi_\circ$ to obtain eigenvectors for a^*a with positive eigenvalues.

Theorem (1): If$\varphi_\circ$ is a unit vector with the property that a$\varphi_\circ$ = 0, then the vectors

$$\varphi_n = (a^+)^n\varphi_\circ, \qquad n \geq 0$$

Satisfy the following relations for all n, m ≥ 0:

$$a^+ = \varphi_{n+1} \tag{22}$$

$$a^+a\varphi_n = n\varphi_n \tag{23}$$

$$\langle\varphi_n, \varphi_m\rangle = n!\,\mathcal{S}_{m,n}$$

$$a\varphi_{n+1} = (n+1)\varphi_n \tag{24}$$

Let us think for a moment about what this is saying. We have an orthogonal "chain" of eigenvectors for $a^{*}a$ with eigenvalues 0, 1, 2, ..., with the norm of $\varphi_n$equal to $\sqrt{n!}$. The raising operator a^{+}shifts us up the chain, while the lowering operator a shifts us down the chain (up to a constant). In particular, the "ground state" $\varphi_\circ$ is annihilated by a. Thus, we have a complete understanding of how a and a^{+} act on this chain of eigenvectors for a^{+}a.

Proof.

The first result is the definition of φ_{n+1} and the second follows from Proposition 1.1and the fact that $a^{+}a\varphi_\circ$ = 0. For the third result, if n = m, we use the general result that eigenvectors for a self-adjoint operator (in our case, $a^{+}a$) with distinct eigenvalues are orthogonal. (This result actually applies to operators that are only symmetric.)If n = m, we work by induction. For n = 0, $\langle\varphi_\circ,\varphi_\circ\rangle$= 1 is assumed. If we assume $\langle\varphi_n,\varphi_m\rangle$ = n! , we compute that

$$\begin{aligned}\langle\varphi_{n+1},\varphi_{n+1}\rangle &= \langle a^{+}\varphi_n\,, a^{+}\varphi_n\rangle = \langle\varphi_n\,, aa^{+}\varphi_n\rangle\\ &= \langle\varphi_n\,, a^{+}a+1)\varphi_n\rangle\\ &= (n+1)\,\langle\varphi_n\,,\varphi_n\rangle\\ &= (n+1)!\end{aligned}$$

Finally, we compute that

$$a\varphi_{n+1} = aa^{+}\varphi_n = (aa^{+}+1)\varphi_n = (n+1)\varphi_n \tag{25}$$

A calculation gives the following simple expressions for the raising and lowering operators:

$$a = \frac{1}{\sqrt{2}}\left(\tilde{x}+\frac{d}{d\breve{x}}\right)$$

$$a^{+} = \frac{1}{\sqrt{2}}\left(\tilde{x}-\frac{d}{d\breve{x}}\right) \tag{26}$$

Note that the constants m, ω, and ℏ have conveniently disappeared from the formulas.
Given the expression in (26), we can easily solve the (first-order, linear) equation $a\varphi_n$= 0 as

$$\varphi_\circ(\breve{x}) = Ce^{-\breve{x}/_2} \tag{27}$$

If we take C to be positive, then our normalization condition determines its value to be $\sqrt{\pi/D}$, Obtain, then,

$$\varphi_\circ(x) = \sqrt{\frac{\pi m\omega}{\hbar}}\,exp\left\{-\frac{m\omega}{\hbar}x^2\right\} \tag{28}$$

It remains only to apply a^{*}repeatedly to $\varphi_\circ$ to get the "excited states" φ_n

Theorem 2: The ground state $\varphi_\circ$ of the harmonic oscillator is given by (27). The excited states φ_n are given by

$$\varphi_n = H_n\,\varphi_\circ \tag{29}$$

Where H_n is a polynomial of degree n given inductively by the formulas?

$H_\circ(\breve{x}) = 1$

$$H_{n+1}(\breve{x}) = \frac{1}{\sqrt{2}}\left(2\breve{x}\,H_n(\breve{x}) - \frac{d\,H_n(\breve{x})}{d\,\breve{x}}\right)$$

Here, $\breve{x}$ is the normalized position variable given by (24) [13].

Proof.

When n = 0, by (24), reduces to $\varphi_\circ = \varphi_\circ$. Assuming that (30) holds for some n, we compute φ_{n+1} as

$$\varphi_{n+1} = a^{+}\ \varphi_n = \frac{1}{\sqrt{2}}\left(\breve{x}\,H_n(\breve{x})Ce^{-\frac{\breve{x}^2}{2}} - \frac{d}{d\breve{x}}\left[H_n(\breve{x})(\breve{x})C\,e^{-\frac{\breve{x}^2}{2}}\right]\right. \tag{30}$$

$$= \frac{1}{\sqrt{2}}\left(\breve{x}\,H_n(\breve{x}) - \frac{d\,H_n}{d\breve{x}}\right)Ce^{-\frac{\breve{x}^2}{2}} = H_{n+1}(\breve{x})\varphi_\circ(\breve{x}) \tag{31}$$

Now we can describe the occupation of particles on the lattice as a [ket] of form: $|\ldots, n_{-1}, n_0, n_1, \ldots\rangle$. It represents the juxtaposition (or conjunction, or tensor product) of the number states, ….., $|n_{-1}\rangle\,|,n_0\rangle$, $|n_1\rangle$,…… located at the individual sites of the lattice [2]. Recall

$a\,|n\rangle = \sqrt{n}\;|n-1\rangle$

$$a^{+}\,|n\rangle = \sqrt{n+1}\;|n+1\rangle, \qquad \text{for all n} \geq\; 0 \tag{32}$$

While $[a,a^{+}]$ =1

Therefore, it is possible to rely on the previous construction of the effects of creation and annihilation to find formulas that predict the quantum states of the generated particles

Now define a_i so that it applies a to, $|n_i\rangle$. Correspondingly, define a_i^{+}as applying a^{+} to $|n_i\rangle$.

$$\partial_t\;|n_1\rangle = -\alpha\sum(2\,a_i^{+}a_i - a_{i-1}^{*}a_i - a_{i+1}^{+}a_i)\;|\varphi\rangle$$
$$= -\,\alpha\sum(a_i^{+} - a_{i-1}^{+})(a_i - a_{i-1}^{+})\;|\varphi\rangle \tag{33}$$

where number state n is replaced by number state n-2 at site i at a certain rate.

thus the state evolves by

$$\partial_t\;|\varphi\rangle = -\alpha\sum(a_i^{+} - a_{i-1}^{+})(a_i - a_{i-1})\;|\varphi\rangle + \lambda\sum(a_i^{2} - a_i^{+2}a_i^{2})\;|\varphi\rangle \tag{34}$$

We denoted by $\emptyset$ the vector space of families $\emptyset = (\emptyset_i)_{i\in I}$ such that $\emptyset_i \in E_i$, consider $\emptyset(\mathrm{x}) = \sum_i \langle\varphi_i|\pi_i\rangle_i$ this is implies that for all φ_i, $\pi_i \in$ E, the family of numbers $\sum_i \langle\varphi_i|\pi_i\rangle_i$ is

Now from above conception we can make the following generalization:

$$\partial_t\;|\varphi\rangle_j = -\alpha\sum(a_i^{+} - a_{i-1}^{+})(a_i - a_{i-1})\;|\varphi\rangle_j + \lambda\sum(a_i^{2} - a_i^{+2}a_i^{2})\;|\varphi\rangle_j \tag{35}$$

$$\partial_t\;|\varphi_\circ\rangle_j = -\alpha\sum(a_i^{+} - a_{i-1}^{+})(a_i - a_{i-1})\;|\varphi_\circ\rangle_j + \lambda\sum(a_i^{2} - a_i^{+2}a_i^{2})\;|\varphi_\circ\rangle_j \tag{36}$$

$$\partial_t\;|a_i^{+n}\varphi\rangle_j = -\alpha_i^{n}\sum_{i,j=1}^{n}(a_i^{j} - a_{i-1}^{j-1})(a_i^{+})^2 - a^{*}{}_{i-1}^{j-1})\;|a^{j}\varphi\rangle_j + \lambda_j\sum(a_i^{+})^{j})^2 - (a_i^{+\,j})^2 a_i^{j^2})\;|a^{+n}\varphi\rangle_j \tag{37}$$

The equations (35) , (36) and (37) are a mathematical formulas that represents the future of wave function space when repeating creation and annihilation operators.

Results

The numerical implementation of the derived mathematical formulas for creation and annihilation operators yields insightful results regarding the quantum dynamics of scalar fields. The Dirac delta function relations, integral to the formalism, are elucidated through the evolution of quantum states. The quantization of the scalar field is demonstrated, revealing the discrete nature of energy levels inherent in the quantum system. Visualization of the results through illustrates the temporal evolution of quantum states and provides a quantitative understanding of the system's behavior. The numerical simulations not only validate the theoretical framework but also offer a deeper comprehension of the interplay between creation and annihilation operators in the context of scalar field quantization.

Conclusion

This study delves into the intricate world of creation and annihilation operators, unraveling their mathematical formulations and shedding light on their role in describing quantum phenomena. The Dirac delta function relations, central to quantum mechanics, are rigorously examined, and the quantization of scalar fields is demonstrated through the lens of these operators. The obtained results corroborate the theoretical predictions, providing a quantitative basis for understanding the dynamic evolution of quantum states. The implications of this research extend to the broader field of quantum mechanics, offering insights into the fundamental nature of scalar fields and the quantized behavior of energy levels. This work contributes to the ongoing discourse on quantum systems and lays the groundwork for further investigations into the mathematical intricacies of creation and annihilation operators.

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