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Some Approximation Properties for the Reverse Order of q-Beta Functions of Second kind

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Abstract: In this paper we propose a new type of variant, $\mathbb{Q}$-variant of *q*-Beta functions of second kind and consider some of its approximation properties. $\mathbb{Q} \in (0,1]$ is known to be the reverse order of *q* in *q*-calculus. We also establish a relation between the generalized beta and gamma functions using some identities. As an application, we propose the $\mathbb{Q}$-Baskakov Durrmeyer operators and get estimates their moments, weighted approximation result and quantitative approximation result.
Keywords: $\mathbb{Q}$-Baskakov-Durrmeyer operators, $\mathbb{Q}$-Beta function of second kind, $\mathbb{Q}$-gamma function, Quantitative approximation, Weighted approximation.
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1. Preliminary

The study of quantum calculus (*q*-calculus) and its results provided the widest scope in approximation theory after the year 1995 {see [2],[3],[7],12]}. Several results through several generalizations for several operators were obtained. Further there arise an extension of *q*-calculus to post-quantum calculus i.e. (*p*, *q*)-calculus [1], [4]. Several results are obtained for these types of operators [8]-[11]. Now we propose new variant, "$\mathbb{Q}$-variant" the reverse order of *q* in *q*-calculus. It was firstly introduced by Garg [6] for $\mathbb{Q}$-Baskakov Durrmeyer operators and Basic notations are mentioned in this paper. For convenience to the readers, we mention here too:

$$[n]_{\mathbb{Q}} = \frac{\mathbb{Q}^n - 1}{\mathbb{Q} - 1}, \qquad n \in N$$

$$[n]_{\mathbb{Q}} = \mathbb{Q}^{n-1}[n]_{1/\mathbb{Q}}, \qquad n \in N$$

$$[n]_{\mathbb{Q}}! = \prod_{k=1}^{n}[k]_{\mathbb{Q}}, \quad n \geq 1, \qquad [0]_{\mathbb{Q}}! = 1$$

$$\begin{bmatrix} n \\ k \end{bmatrix}_{\mathbb{Q}} = \frac{[n]_{\mathbb{Q}}!}{[k]_{\mathbb{Q}}!.\,[n-k]_{\mathbb{Q}}!}, \qquad 0 \leq k \leq n$$

$$D_{\mathbb{Q}} f(x) = \frac{f(\mathbb{Q}x) - f(x)}{(\mathbb{Q}-1)x}, \quad x \neq 0 \ \ \& \ \ D_{\mathbb{Q}} f(0) = f'(0)$$

$$(x \oplus y)_{\mathbb{Q}}^n = (x+y)(\mathbb{Q}x+y)(\mathbb{Q}^2x+y)\ldots\ldots(\mathbb{Q}^{n-1}x+y),$$

$$(x \ominus y)_{\mathbb{Q}}^n = (x-y)(\mathbb{Q}x-y)(\mathbb{Q}^2x-y)\ldots\ldots(\mathbb{Q}^{n-1}x-y),$$

$$B_{\mathbb{Q}}(m,n) = \mathbb{Q}^{\frac{n}{2}} \int_0^{\infty/A} \frac{x^{m-1}}{(1 \oplus x)_{\mathbb{Q}}^{m+n}} d_{\mathbb{Q}}x, \qquad m,n \in N$$

$$\Gamma_{\mathbb{Q}}(n+1) = \frac{(\mathbb{Q} \ominus 1)_{\mathbb{Q}}^n}{(\mathbb{Q}-1)^n} = [n]_{\mathbb{Q}}!, \qquad 0 < \mathbb{Q} < 1$$

Preposition 1: The $\mathbb{Q}$-derivative of the product is defined as-

$$D_{\mathbb{Q}}(f(x)g(x)) = f(\mathbb{Q}x)D_{\mathbb{Q}}g(x) + g(x)D_{\mathbb{Q}}f(x)$$
$$D_{\mathbb{Q}}(f(x)g(x)) = g(\mathbb{Q}x)D_{\mathbb{Q}}f(x) + f(x)D_{\mathbb{Q}}f(x)$$

Preposition 2: The $\mathbb{Q}$-integration by parts is defined as-

$$\int_a^b g(x)D_{\mathbb{Q}}h(x)d_{\mathbb{Q}}x = g(b)h(b) - g(a)h(a) - \int_a^b h(x)D_{\mathbb{Q}}g(x)d_{\mathbb{Q}}x$$

In the present paper, we propose the $\mathbb{Q}$-analogue of genuine Baskakov operators using beta function of second kind and estimate its some approximation properties including asymptotic formula and convergence in terms of modulus of continuity.

In 2023, S. Garg [6] introduced a new type of variant $\mathbb{Q}$-analogue of genuine Baskakov-Durrmeyer operators for $[0,\infty)$ and discussed some of its approximation properties. Now in this paper we propose the $\mathbb{Q}$-variant of q-Beta functions of second kind and consider some of its approximation properties. We also establish a relation between the generalized beta and gamma functions using some identities. As an application, we also propose the $\mathbb{Q}$-Baskakov Durrmeyer operators and get estimates of their moments and some results for weighted approximation and quantitative approximation.

2. $\mathbb{Q}$ -Beta functions of second kind

Let us define $\mathbb{Q}$ -Beta functions of second kind as

$$B_{\mathbb{Q}}(m,n) = \int_0^\infty \frac{x^{m-1}}{(1\oplus \mathbb{Q}x)_{\mathbb{Q}}^{m+n}} d_{\mathbb{Q}}x, \qquad m,n \in N$$

Theorem: The relation between $\mathbb{Q}$ -Beta and $\mathbb{Q}$ -Gamma functions can be defined as-

$$B_{\mathbb{Q}}(m,n) = \mathbb{Q}^{-\frac{m(m+1)}{2}} \frac{\Gamma_{\mathbb{Q}}(m)\Gamma_{\mathbb{Q}}(n)}{\Gamma_{\mathbb{Q}}(m+n)}$$

Proof: We know that

$$D_{\mathbb{Q}}\left(\frac{1}{(1\oplus x)_{\mathbb{Q}}^{n}}\right) = \frac{-\mathbb{Q}[n]_{\mathbb{Q}}}{(1\oplus \mathbb{Q}x)_{\mathbb{Q}}^{n+1}}$$

Also if we choose $f(x)$ & $g(x)$ such that $f(x) = x^m$ & $g(x) = \frac{-1}{\mathbb{Q}[m+n]_{\mathbb{Q}}(1\oplus x)_{\mathbb{Q}}^{m+n}}$ then we have:

$$B_{\mathbb{Q}}(m+1,n) = \int_0^\infty \frac{x^m}{(1\oplus \mathbb{Q}x)_{\mathbb{Q}}^{m+n+1}} d_{\mathbb{Q}}x$$

$$= \frac{-\mathbb{Q}^{-m}}{\mathbb{Q}[m+n]_{\mathbb{Q}}} \int_0^\infty (\mathbb{Q}x)^m D_{\mathbb{Q}} \frac{1}{(1\oplus x)_{\mathbb{Q}}^{m+n}} d_{\mathbb{Q}}x$$

$$= \frac{\mathbb{Q}^{-m}}{\mathbb{Q}[m+n]_{\mathbb{Q}}} \int_0^\infty D_{\mathbb{Q}}x^m \frac{1}{(1\oplus x)_{\mathbb{Q}}^{m+n}} d_{\mathbb{Q}}x$$

$$= \frac{\mathbb{Q}^{-m-1}[m]_{\mathbb{Q}}}{[m+n]_{\mathbb{Q}}} \int_0^\infty \frac{x^{m-1}}{(1\oplus \mathbb{Q}x)_{\mathbb{Q}}^{m+n}} d_{\mathbb{Q}}x$$

$$= \frac{\mathbb{Q}^{-1}[m]_{\mathbb{Q}}}{\mathbb{Q}^m[m+n]_{\mathbb{Q}}} \int_0^\infty \frac{x^{m-1}}{(1\oplus \mathbb{Q}x)_{\mathbb{Q}}^{m+n}} d_{\mathbb{Q}}x$$

$$= \frac{\mathbb{Q}^{-1}[m]_{\mathbb{Q}}}{\mathbb{Q}^m[m+n]_{\mathbb{Q}}} B_{\mathbb{Q}}(m,n).$$

$$\therefore B_{\mathbb{Q}}(1,n) = \int_0^\infty \frac{1}{(1 \oplus \mathbb{Q}x)_{\mathbb{Q}}^n} d_{\mathbb{Q}}x = \frac{-1}{\mathbb{Q}[n]_{\mathbb{Q}}} \int_0^\infty D_{\mathbb{Q}} \frac{1}{(1 \oplus x)_{\mathbb{Q}}^n} d_{\mathbb{Q}}x = \frac{1}{\mathbb{Q}[n]_{\mathbb{Q}}}$$

$$B_{\mathbb{Q}}(m,n) = \frac{\mathbb{Q}^{-1}[m-1]_{\mathbb{Q}}}{\mathbb{Q}^{m-1}[m+n-1]_{\mathbb{Q}}} B_{\mathbb{Q}}(m-1,n)$$

$$= \frac{\mathbb{Q}^{-1}[m-1]_{\mathbb{Q}}}{\mathbb{Q}^{m-1}[m+n-1]_{\mathbb{Q}}} \frac{\mathbb{Q}^{-1}[m-2]_{\mathbb{Q}}}{\mathbb{Q}^{m-2}[m+n-2]_{\mathbb{Q}}} B_{\mathbb{Q}}(m-2,n)$$

$$= \frac{\mathbb{Q}^{-1}[m-1]_{\mathbb{Q}}}{\mathbb{Q}^{m-1}[m+n-1]_{\mathbb{Q}}} \frac{\mathbb{Q}^{-1}[m-2]_{\mathbb{Q}}}{\mathbb{Q}^{m-2}[m+n-2]_{\mathbb{Q}}} \cdots\cdots\cdots\cdots \frac{\mathbb{Q}^{-1}}{\mathbb{Q}[n+1]_{\mathbb{Q}}} B_{\mathbb{Q}}(1,n)$$

$$= \frac{\mathbb{Q}^{-1}[m-1]_{\mathbb{Q}}}{\mathbb{Q}^{m-1}[m+n-1]_{\mathbb{Q}}} \frac{\mathbb{Q}^{-1}[m-2]_{\mathbb{Q}}}{\mathbb{Q}^{m-2}[m+n-2]_{\mathbb{Q}}} \cdots\cdots\cdots\cdots \frac{\mathbb{Q}^{-1}}{\mathbb{Q}[n+1]_{\mathbb{Q}}} \cdot \frac{-1}{\mathbb{Q}[n]_{\mathbb{Q}}}$$

$$= \mathbb{Q}^{-\frac{m(m+1)}{2}} \frac{\Gamma_{\mathbb{Q}}(m)\Gamma_{\mathbb{Q}}(n)}{\Gamma_{\mathbb{Q}}(m+n)}$$

3. $\mathbb{Q}$ −Baskakov Durrmeyer Operators

The $\mathbb{Q}$-analogue of Baskakov operators for $x \in [0,\infty)$ & $0 < \mathbb{Q} \le 1$ can be defined as:

$$B_n^{\mathbb{Q}}(f,x) = \sum_{v=0}^{\infty} p_{n,v}^{\mathbb{Q}}(x)\, f\left(\frac{\mathbb{Q}^{n-1}[v]_{\mathbb{Q}}}{[n]_{\mathbb{Q}}}\right) \qquad \dots (3.1)$$

where

$$p_{n,v}^{\mathbb{Q}}(x) = \begin{bmatrix} n+v-1 \\ v \end{bmatrix}_{\mathbb{Q}} \mathbb{Q}^{v+n(n-1)/2} \frac{x^v}{(1 \oplus x)_{\mathbb{Q}}^{n+v}} \qquad \dots (3.2)$$

In case $\mathbb{Q} = 1$, we get the well-known Baskakov operators and Baskakov Basis function from (3.1) and (3.2) respectively. Also, from here we can easily compute the following results:

$$B_n^{\mathbb{Q}}(1,x) = 1, \qquad B_n^{\mathbb{Q}}(t,x) = x, \qquad B_n^{\mathbb{Q}}(t^2,x) = x^2 + \frac{\mathbb{Q}^{n-1}x}{[n]_{\mathbb{Q}}}(1+\mathbb{Q}x)$$

Now, using $\mathbb{Q}$-Beta function of second kind, we can propose the $\mathbb{Q}$-analogue of Baskakov Durrmeyer operators for $x \in [0,\infty)$ & $0 < \mathbb{Q} \le 1$, as

$$M_n^{\mathbb{Q}}(f,x) = [n-1]_{\mathbb{Q}} \sum_{v=0}^{\infty} p_{n,v}^{\mathbb{Q}}(x) \mathbb{Q}^{\frac{(v+1)(v+2)}{2}} \int_0^\infty \begin{bmatrix} n+v-1 \\ v \end{bmatrix}_{\mathbb{Q}} \frac{t^v}{(1 \oplus \mathbb{Q}t)_{\mathbb{Q}}^{n+v}} f(\mathbb{Q}^v t) d_{\mathbb{Q}}t \qquad \dots (3.3)$$

where $p_{n,v}^{\mathbb{Q}}(x)$ is defined as above.

4. Auxiliary Results

In this section, we give some Lemmas to get main results.

Lemma:1 For $x \in [0,\infty)$ & $0 < \mathbb{Q} \le 1$, we have

$$M_n^{\mathbb{Q}}(1,x) = 1, \qquad M_n^{\mathbb{Q}}(t,x) = \frac{1}{\mathbb{Q}^2[n-2]_{\mathbb{Q}}} + \frac{1}{\mathbb{Q}^n}\left(\frac{[2]_{\mathbb{Q}}}{[n-2]_{\mathbb{Q}}} + \mathbb{Q}^2\right)x$$

$$M_n^{\mathbb{Q}}(t^2,x) = \frac{[2]_{\mathbb{Q}}}{[n-2]_{\mathbb{Q}}[n-3]_{\mathbb{Q}}} + \left[\frac{\{\mathbb{Q}^5(1+2\mathbb{Q})+1\}[3]_{\mathbb{Q}}}{\mathbb{Q}^6[n-2]_{\mathbb{Q}}[n-3]_{\mathbb{Q}}} + \frac{\mathbb{Q}^5(1+2\mathbb{Q})+1}{\mathbb{Q}^{n+3}[n-2]_{\mathbb{Q}}}\right]x$$

$$+ \left[\frac{\mathbb{Q}^2+\mathbb{Q}+1}{\mathbb{Q}^{n+9}[n-2]_{\mathbb{Q}}} + \frac{[3]_{\mathbb{Q}}}{\mathbb{Q}^{n+10}[n-3]_{\mathbb{Q}}} + \frac{\left(\mathbb{Q}^{n+2}[3]_{\mathbb{Q}} + [2]_{\mathbb{Q}}[3]_{\mathbb{Q}}\right)}{\mathbb{Q}^{n+12}[n-2]_{\mathbb{Q}}[n-3]_{\mathbb{Q}}} + \frac{1}{\mathbb{Q}^{2n+7}}\right]x^2$$

Proof: From (3.3), we have

$$M_n^{\mathbb{Q}}(1,x) = [n-1]_{\mathbb{Q}} \sum_{v=0}^{\infty} p_{n,v}^{\mathbb{Q}}(x) \mathbb{Q}^{\frac{(v+1)(v+2)}{2}} \int_0^{\infty} \begin{bmatrix} n+v-1 \\ v \end{bmatrix}_{\mathbb{Q}} \frac{t^v}{(1 \oplus \mathbb{Q}t)_{\mathbb{Q}}^{n+v}} d_{\mathbb{Q}}t$$

$$= [n-1]_{\mathbb{Q}} \sum_{v=0}^{\infty} p_{n,v}^{\mathbb{Q}}(x) \mathbb{Q}^{\frac{(v+1)(v+2)}{2}} \begin{bmatrix} n+v-1 \\ v \end{bmatrix}_{\mathbb{Q}} B_{\mathbb{Q}}(v+1, n-1)$$

$$= B_n^{\mathbb{Q}}(1,x) = 1$$

Now using identity $[n+1]_{\mathbb{Q}} = 1 + \mathbb{Q}[n]_{\mathbb{Q}}$

$$M_n^{\mathbb{Q}}(t,x) = [n-1]_{\mathbb{Q}} \sum_{v=0}^{\infty} p_{n,v}^{\mathbb{Q}}(x) \mathbb{Q}^{\frac{(v+1)(v+2)}{2}} \int_0^{\infty} \begin{bmatrix} n+v-1 \\ v \end{bmatrix}_{\mathbb{Q}} \frac{t^{v+1}\mathbb{Q}^v}{(1 \oplus \mathbb{Q}t)_{\mathbb{Q}}^{n+v}} d_{\mathbb{Q}}t$$

$$= [n-1]_{\mathbb{Q}} \sum_{v=0}^{\infty} p_{n,v}^{\mathbb{Q}}(x) \mathbb{Q}^{\frac{(v+1)(v+2)}{2}} \begin{bmatrix} n+v-1 \\ v \end{bmatrix}_{\mathbb{Q}} \mathbb{Q}^v B_{\mathbb{Q}}(v+2, n-2)$$

$$= [n-1]_{\mathbb{Q}} \sum_{v=0}^{\infty} p_{n,v}^{\mathbb{Q}}(x) \mathbb{Q}^{\frac{(v+1)(v+2)}{2}} \frac{[n+v-1]_Q!}{[v]_Q!\,[n-1]_Q!} \mathbb{Q}^v \mathbb{Q}^{-(v+2)(v+3)/2} \frac{\Gamma_{\mathbb{Q}}(v+2).\Gamma_{\mathbb{Q}}(n-2)}{\Gamma_{\mathbb{Q}}(n+v)}$$

$$= \sum_{v=0}^{\infty} p_{n,v}^{\mathbb{Q}}(x) \mathbb{Q}^{-2} \frac{[v+1]_{\mathbb{Q}}}{[n-2]_{\mathbb{Q}}}$$

$$= \frac{1}{\mathbb{Q}^2[n-2]_{\mathbb{Q}}} \sum_{v=0}^{\infty} p_{n,v}^{\mathbb{Q}}(x) \{1 + \mathbb{Q}[v]_{\mathbb{Q}}\}$$

$$= \frac{1}{\mathbb{Q}^2[n-2]_{\mathbb{Q}}} B_n^{\mathbb{Q}}(1,x) + \frac{[n]_{\mathbb{Q}}}{\mathbb{Q}^n[n-2]_{\mathbb{Q}}} B_n^{\mathbb{Q}}(t,x) = \frac{\mathbf{1}}{\mathbb{Q}^2[\boldsymbol{n}-\mathbf{2}]_{\mathbb{Q}}} + \frac{\mathbf{1}}{\mathbb{Q}^{\boldsymbol{n}}} \left(\frac{[\mathbf{2}]_{\mathbb{Q}}}{[\boldsymbol{n}-\mathbf{2}]_{\mathbb{Q}}} + \mathbb{Q}^{\mathbf{2}} \right) \boldsymbol{x}$$

Now, using identity $[n+2]_{\mathbb{Q}} = 1 + \mathbb{Q} + \mathbb{Q}^2[n]_{\mathbb{Q}}$, we have the required result. Proceeding it:

$$M_n^{\mathbb{Q}}(t^2,x) = [n-1]_{\mathbb{Q}} \sum_{v=0}^{\infty} p_{n,v}^{\mathbb{Q}}(x) \mathbb{Q}^{\frac{(v+1)(v+2)}{2}} \int_0^{\infty} \begin{bmatrix} n+v-1 \\ v \end{bmatrix}_{\mathbb{Q}} \frac{t^{v+2}\mathbb{Q}^{2v}}{(1 \oplus \mathbb{Q}t)_{\mathbb{Q}}^{n+v}} d_{\mathbb{Q}}t$$

$$= [n-1]_{\mathbb{Q}} \sum_{v=0}^{\infty} p_{n,v}^{\mathbb{Q}}(x) \mathbb{Q}^{\frac{(v+1)(v+2)}{2}} \begin{bmatrix} n+v-1 \\ v \end{bmatrix}_{\mathbb{Q}} \mathbb{Q}^{2v} B_{\mathbb{Q}}(v+3, n-3)$$

$$= [n-1]_{\mathbb{Q}} \sum_{v=0}^{\infty} p_{n,v}^{\mathbb{Q}}(x) \mathbb{Q}^{\frac{(v+1)(v+2)}{2}} \frac{[n+v-1]_Q!}{[v]_Q!\,[n-1]_Q!} \mathbb{Q}^{2v} \mathbb{Q}^{-(v+3)(v+4)/2} \frac{\Gamma_{\mathbb{Q}}(v+3).\Gamma_{\mathbb{Q}}(n-3)}{\Gamma_{\mathbb{Q}}(n+v)}$$

$$= \sum_{v=0}^{\infty} p_{n,v}^{\mathbb{Q}}(x) \mathbb{Q}^{-5} \frac{[v+2]_{\mathbb{Q}}[v+1]_{\mathbb{Q}}}{[n-2]_{\mathbb{Q}}[n-3]_{\mathbb{Q}}}$$

$$= \sum_{v=0}^{\infty} p_{n,v}^{\mathbb{Q}}(x) \mathbb{Q}^{-5} \frac{\{1 + \mathbb{Q} + \mathbb{Q}^2[v]_{\mathbb{Q}}\}\{1 + \mathbb{Q}[v]_{\mathbb{Q}}\}}{[n-2]_{\mathbb{Q}}[n-3]_{\mathbb{Q}}}$$

$$= \sum_{v=0}^{\infty} p_{n,v}^{\mathbb{Q}}(x) \mathbb{Q}^{-5} \frac{\{\mathbb{Q}^3[v]_{\mathbb{Q}}{}^2 + (\mathbb{Q}[2]_{\mathbb{Q}} + \mathbb{Q}^2)[v]_{\mathbb{Q}} + [2]_{\mathbb{Q}}\}}{[n-2]_{\mathbb{Q}}[n-3]_{\mathbb{Q}}}$$

$$= \frac{1}{[n-2]_{\mathbb{Q}}[n-3]_{\mathbb{Q}}} \sum_{v=0}^{\infty} p_{n,v}^{\mathbb{Q}}(x) \begin{bmatrix} \left(\mathbb{Q}^{n-1}[v]_{\mathbb{Q}}\right)^2 \mathbb{Q}^{-7-2n} + \left([2]_{\mathbb{Q}} + \mathbb{Q}\right) \\ \left(\mathbb{Q}^{n-1}[v]_{\mathbb{Q}}\right)\mathbb{Q}^{-3-n} + [2]_{\mathbb{Q}} \end{bmatrix}$$

$$= \frac{\mathbb{Q}^{-7-2n}[n]_{\mathbb{Q}}{}^2}{[n-2]_{\mathbb{Q}}[n-3]_{\mathbb{Q}}} B_n^{\mathbb{Q}}(t^2,x) + \frac{\mathbb{Q}^{-3-n}([2]_{\mathbb{Q}} + \mathbb{Q})[n]_{\mathbb{Q}}}{[n-2]_{\mathbb{Q}}[n-3]_{\mathbb{Q}}} B_n^{\mathbb{Q}}(t,x) + \frac{[2]_{\mathbb{Q}}}{[n-2]_{\mathbb{Q}}[n-3]_{\mathbb{Q}}} B_n^{\mathbb{Q}}(1,x)$$

$$= \frac{\mathbb{Q}^{-7-2n}{[n]_{\mathbb{Q}}}^2}{[n-2]_{\mathbb{Q}}[n-3]_{\mathbb{Q}}}\left\{x^2 + \frac{\mathbb{Q}^{n-1}x}{[n]_{\mathbb{Q}}}(1+\mathbb{Q}x)\right\} + \frac{\mathbb{Q}^{-3-n}([2]_{\mathbb{Q}}+\mathbb{Q})[n]_{\mathbb{Q}}}{[n-2]_{\mathbb{Q}}[n-3]_{\mathbb{Q}}}x + \frac{[2]_{\mathbb{Q}}}{[n-2]_{\mathbb{Q}}[n-3]_{\mathbb{Q}}}.1$$

$$= \frac{[2]_{\mathbb{Q}}}{[n-2]_{\mathbb{Q}}[n-3]_{\mathbb{Q}}} + \left\{\frac{\mathbb{Q}^{-3-n}([2]_{\mathbb{Q}}+\mathbb{Q})[n]_{\mathbb{Q}}}{[n-2]_{\mathbb{Q}}[n-3]_{\mathbb{Q}}} + \frac{\mathbb{Q}^{-7-2n}{[n]_{\mathbb{Q}}}^2}{[n-2]_{\mathbb{Q}}[n-3]_{\mathbb{Q}}}\frac{\mathbb{Q}^{n-1}}{[n]_{\mathbb{Q}}}\right\}x + \left\{\frac{\mathbb{Q}^{-7-2n}{[n]_{\mathbb{Q}}}^2}{[n-2]_{\mathbb{Q}}[n-3]_{\mathbb{Q}}}\left(1+\frac{\mathbb{Q}^n}{[n]_{\mathbb{Q}}}\right)x^2\right\}$$

$$= \frac{[2]_{\mathbb{Q}}}{[n-2]_{\mathbb{Q}}[n-3]_{\mathbb{Q}}} + \left[\frac{\{\mathbb{Q}^5(1+2\mathbb{Q})+1\}[3]_{\mathbb{Q}}}{\mathbb{Q}^6[n-2]_{\mathbb{Q}}[n-3]_{\mathbb{Q}}} + \frac{\mathbb{Q}^5(1+2\mathbb{Q})+1}{\mathbb{Q}^{n+3}[n-2]_{\mathbb{Q}}}\right]x + \left[\frac{\mathbb{Q}^2+\mathbb{Q}+1}{\mathbb{Q}^{n+9}[n-2]_{\mathbb{Q}}} + \frac{[3]_{\mathbb{Q}}}{\mathbb{Q}^{n+10}[n-3]_{\mathbb{Q}}} + \frac{(\mathbb{Q}^{n+2}[3]_{\mathbb{Q}}+[2]_{\mathbb{Q}}[3]_{\mathbb{Q}})}{\mathbb{Q}^{n+12}[n-2]_{\mathbb{Q}}[n-3]_{\mathbb{Q}}} + \frac{1}{\mathbb{Q}^{2n+7}}\right]x^2$$

Hence the proof of the lemma by using identity $[n+3]_{\mathbb{Q}} = 1+\mathbb{Q}+\mathbb{Q}^2+\mathbb{Q}^3[n]_{\mathbb{Q}}$.

5. Main Results

In this section, we consider some approximation results, weighted approximation and quantitative approximation results.

5.1 Weighted Approximation

To prove the weighted approximation theorem, we have first need of a class of functions on the interval $[0,\infty)$, defined as:

Let $H_{x^2}[0,\infty)$ be the set of all functions f defined on $[0,\infty)$ satisfying the condition $|f(x)| \le M_f(1+x^2)$, where M_f is a constant depending only on f. By $C_{x^2}[0,\infty)$, we denote the subspace of all continuous functions belonging to $H_{x^2}[0,\infty)$. Also, let $C^*_{x^2}[0,\infty)$ be the subspace of all functions $f \in C_{x^2}[0,\infty)$, for which $\lim_{|x|\to\infty}\frac{f(x)}{1+x^2}$ is finite. The norm on $C^*_{x^2}[0,\infty)$ is $\|f\|_{x^2} = \sup_{x\in[0,\infty)}\frac{|f(x)|}{1+x^2}$.

Theorem 1: Let $\mathbb{Q}=\mathbb{Q}_n$ satisfying $0<\mathbb{Q}\le 1$ and for sufficiently large n, $\mathbb{Q}_n \to 1$ and $\mathbb{Q}_n^n \to 1$. For each $f \in C^*_{x^2}[0,\infty)$, we have

$$\lim_{n\to\infty}\left\|M_n^{\mathbb{Q}_n}(f(t),x)-f(x)\right\|_{x^2}=0$$

Proof: From previous study, we have noted that using several operators for this theorem, it is sufficient to verify the following three conditions:

$$\lim_{n\to\infty}\left\|M_n^{\mathbb{Q}_n}(t^m,x)-x^m\right\|_{x^2}=0, \qquad m=0,1,2.$$

Hence for the condition $m=0$, the proof of theorem is obvious for all n. Considering for $m=1$ and $n>2$, we get:

$$\left\|M_n^{\mathbb{Q}_n}(t,x)-x\right\|_{x^2} = \sup_{x\in[0,\infty)}\frac{1}{1+x^2}\left\{\frac{1}{{\mathbb{Q}_n}^2[n-2]_{\mathbb{Q}_n}} + \frac{1}{{\mathbb{Q}_n}^n}\left(\frac{[2]_{\mathbb{Q}_n}}{[n-2]_{\mathbb{Q}_n}}+{\mathbb{Q}_n}^2-1\right)x\right\}$$

$$\le \frac{1}{{\mathbb{Q}_n}^2[n-2]_{\mathbb{Q}_n}}.\sup_{x\in[0,\infty)}\frac{1}{1+x^2} + \frac{1}{{\mathbb{Q}_n}^n}\left(\frac{[2]_{\mathbb{Q}_n}}{[n-2]_{\mathbb{Q}_n}}+{\mathbb{Q}_n}^2-1\right).\sup_{x\in[0,\infty)}\frac{x}{1+x^2}$$

$$\Rightarrow \qquad \lim_{n\to\infty}\left\|M_n^{\mathbb{Q}_n}(t,x)-x\right\|_{x^2}=0$$

Similarly, for $m=2$ and $n>3$, we get:

$$\left\|M_n^{\mathbb{Q}_n}(t^2,x)-x^2\right\|_{x^2} \le \frac{[2]_{\mathbb{Q}_n}}{[n-2]_{\mathbb{Q}_n}[n-3]_{\mathbb{Q}_n}}\sup_{x\in[0,\infty)}\frac{1}{1+x^2} + \left[\frac{\{{\mathbb{Q}_n}^5(1+2\mathbb{Q}_n)+1\}[3]_{\mathbb{Q}_n}}{{\mathbb{Q}_n}^6[n-2]_{\mathbb{Q}_n}[n-3]_{\mathbb{Q}_n}} + \frac{{\mathbb{Q}_n}^5(1+2\mathbb{Q}_n)+1}{{\mathbb{Q}_n}^{n+3}[n-2]_{\mathbb{Q}_n}}\right]\sup_{x\in[0,\infty)}\frac{x}{1+x^2}$$

$$+\left[\frac{{\mathbb{Q}_n}^2+\mathbb{Q}_n+1}{{\mathbb{Q}_n}^{n+9}[n-2]_{\mathbb{Q}_n}} + \frac{[3]_{\mathbb{Q}_n}}{{\mathbb{Q}_n}^{n+10}[n-3]_{\mathbb{Q}_n}} + \frac{({\mathbb{Q}_n}^{n+2}[3]_{\mathbb{Q}_n}+[2]_{\mathbb{Q}_n}[3]_{\mathbb{Q}_n})}{{\mathbb{Q}_n}^{n+12}[n-2]_{\mathbb{Q}_n}[n-3]_{\mathbb{Q}_n}} + \frac{1}{{\mathbb{Q}_n}^{2n+7}} - 1\right]\sup_{x\in[0,\infty)}\frac{x^2}{1+x^2}$$

$$\Rightarrow \qquad \lim_{n\to\infty}\left\|M_n^{\mathbb{Q}_n}(t^2,x)-x^2\right\|_{x^2}=0$$

Hence our theorem is true for all the functions of the type $f(t)=t^m$. In addition to generalize the theorem for all $f\in C_{x^2}^*[0,\infty)$, it is necessary to show that:

$$\lim_{n\to\infty}\left\|M_n^{\mathbb{Q}_n}(f(t),x)-f(x)\right\|_{x^2}=\lim_{n\to\infty}\sup_{x\in[0,\infty)}\frac{\left|M_n^{\mathbb{Q}_n}(f(t),x)-f(x)\right|}{1+x^2}=0$$

Therefore, we consider a fixed point $0<x_0<\infty$ in the interval $[0,\infty)$ such that:

$$\sup_{x\in[0,\infty)}\frac{\left|M_n^{\mathbb{Q}_n}(f(t),x)-f(x)\right|}{1+x^2}\le\sup_{x\le x_0}\frac{\left|M_n^{\mathbb{Q}_n}(f(t),x)-f(x)\right|}{1+x^2}+\sup_{x\ge x_0}\frac{\left|M_n^{\mathbb{Q}_n}(f(t),x)-f(x)\right|}{1+x^2}$$

$$\le\left\|M_n^{\mathbb{Q}_n}(f(t),x)-f(x)\right\|_{C_{x^2}^*[0,x_0)}+\sup_{x\ge x_0}\frac{\left|M_n^{\mathbb{Q}_n}(1+t^2,x)\right|}{1+x^2}+\sup_{x\ge x_0}\frac{|f(x)|}{1+x^2}$$

The first term on RHS tends to zero from well-known Korovkin's theorem for all n. Taking any fixed $x_0>0$ and from above, it is easy to show that the second term also vanishes as $n\to\infty$, and now for the last part, we can choose $x_0>0$ too large to make it small enough to vanish. Thus, we get the desired.

5.2 Quantitative Approximation

Let $C_B[0,\infty)$, denote the space of all real valued continuous and bounded functions on $f\in C_B[0,\infty)$. In this space we consider the norm

$$\|f\|_{C_B[0,\infty)}=\sup_{x\in[0,\infty)}|f(x)|$$

Also, the modulus of continuity [5] of first as well as second order are defined as:

$$\omega_1(f,\delta)=\sup_{\substack{x,u,v\ge0\\|u-v|\le\delta}}|f(x+u)-f(x+v)|$$

$$\omega_2(f,\delta)=\sup_{\substack{x,u,v\ge0\\|u-v|\le\delta}}|f(x+2u)-2f(x+u+v)+f(x+2v)|,\qquad\delta>0$$

Now the Steklov mean function for $f\in C_B[0,\infty)$ is:

$$f_h(x)=\frac{4}{h^2}\int_0^{h/2}\int_0^{h/2}[2f(x+u+v)-f(x+2(u+v))]dudv \qquad \dots(6.1)$$

Obviously $f_h\in C_B[0,\infty)$, we can write

$$f_h(x)-f(x)=\frac{4}{h^2}\int_0^{h/2}\int_0^{h/2}\left[2f(x+u+v)-f\big(x+2(u+v)\big)-f(x)\right]dudv$$

Obviously, we see that:

$$|f_h(x)-f(x)|\le\omega_2(f,h)$$

$$\Rightarrow \qquad \|f_h(x)-f(x)\|_B\le\omega_2(f,h) \qquad \dots(6.2)$$

If f is continuous, $f_h{}'\in C_B[0,\infty)$ and

$$f_h'=\frac{4}{h^2}\left[2\int_0^{h/2}\left\{2f\left(x+v+\frac{h}{2}\right)-f(x+v)\right\}dv-\frac{1}{2}\int_0^{h/2}\{2f(x+2v+h)-f(x+v)\}dv\right]$$

Thus, we have

$$\|f_h'\|_{C_B[0,\infty)}\le\frac{5}{h}\omega_1(f,h) \qquad \dots(6.3)$$

Similarly, $f_h{}''\in C_B[0,\infty)$ and

$$\|f_h''\|_{C_B[0,\infty)}\le\frac{9}{h^2}\omega_2(f,h) \qquad \dots(6.4)$$

Theorem 2. If $\mathbb{Q}\in(0,1]$ and the operators (3.3) maps space C_B into C_B then

$$\left\|M_n^{\mathbb{Q}}(f,x)\right\|_{C_B}\le\|f\|_{C_B}$$

Proof: From operators (3.3) for $\mathbb{Q}\in(0,1]$, we have:

$$\left|M_n^{\mathbb{Q}}(f,x)\right|\le[n-1]_{\mathbb{Q}}\sum_{v=0}^{\infty}p_{n,v}^{\mathbb{Q}}(x)\mathbb{Q}^{\frac{(v+1)(v+2)}{2}}\int_0^{\infty}\begin{bmatrix}n+v-1\\v\end{bmatrix}_{\mathbb{Q}}\frac{t^v}{(1\oplus\mathbb{Q}t)_{\mathbb{Q}}^{n+v}}|f(\mathbb{Q}^v t)|d_{\mathbb{Q}}t$$

$$\leq \sup_{x\in[0,\infty)} |f(x)|\,[n-1]_{\mathbb{Q}} \sum_{v=0}^{\infty} p_{n,v}^{\mathbb{Q}}(x)\mathbb{Q}^{\frac{(v+1)(v+2)}{2}} \int_0^{\infty} \begin{bmatrix} n+v-1 \\ v \end{bmatrix}_{\mathbb{Q}} \frac{t^v}{(1\oplus \mathbb{Q}t)_{\mathbb{Q}}^{n+v}} d_{\mathbb{Q}}t$$

$$\leq \sup_{x\in[0,\infty)} |f(x)|\, M_n^{\mathbb{Q}}(1,x) = \|\boldsymbol{f}\|_{C_B}$$

Hence the proof is completed.

Theorem 3. Let $\mathbb{Q} \in (0,1]$ and the operators (3.3) maps space C_B into C_B then

$$\left|M_n^{\mathbb{Q}}(f,x) - f(x)\right| \leq 5\omega_1\left(f, \frac{1}{\sqrt{[n-2]_{\mathbb{Q}}}}\right)\left\{\frac{\mathbb{Q}^{n-2} + [2]_{\mathbb{Q}}}{\mathbb{Q}^n\sqrt{[n-2]_{\mathbb{Q}}}} + \left(\frac{1}{\mathbb{Q}^{n-2}} - 1\right)\sqrt{[n-2]_{\mathbb{Q}}}x\right\}$$

$$+\frac{9}{2}\omega_2\left(f, \frac{1}{\sqrt{[n-2]_{\mathbb{Q}}}}\right)\left[\begin{array}{l} \dfrac{[2]_{\mathbb{Q}}}{[n-3]_{\mathbb{Q}}} + \left\{\dfrac{\{\mathbb{Q}^5(1+2\mathbb{Q})+1\}[3]_{\mathbb{Q}}}{\mathbb{Q}^6[n-3]_{\mathbb{Q}}} - \dfrac{2\mathbb{Q}^{n+1} - \mathbb{Q}^5(1+2\mathbb{Q}) - 1}{\mathbb{Q}^{n+3}}\right\}x \\ + \left\{\begin{array}{l} \dfrac{\mathbb{Q}^2 + (1-2\mathbb{Q}^9)[2]_{\mathbb{Q}}}{\mathbb{Q}^{n+9}} + \dfrac{(\mathbb{Q}^{n+2} + [2]_{\mathbb{Q}})[3]_{\mathbb{Q}}}{\mathbb{Q}^{n+12}[n-3]_{\mathbb{Q}}} + \\ \left(\dfrac{\mathbb{Q}^{2n+7} - 2\mathbb{Q}^{n+9} + 1}{\mathbb{Q}^{2n+7}} + \dfrac{[3]_{\mathbb{Q}}[n-2]_{\mathbb{Q}}}{\mathbb{Q}^{n+10}[n-3]_{\mathbb{Q}}}\right)[n-2]_{\mathbb{Q}} \end{array}\right\}x^2 \end{array}\right]$$

Proof: Using Steklov mean function f_h defined above (6.1) for $x \geq 0$ & $n \in N$, we have:

$$\left|M_n^{\mathbb{Q}}(f,x) - f(x)\right| \leq M_n^{\mathbb{Q}}(|(f-f_h)(t)|,x) + \left|M_n^{\mathbb{Q}}(f_h(t) - f_h(x),x)\right| + |f_h(x) - f(x)|$$

By relation (6.2) and Theorem 6.1, we can write:

$$M_n^{\mathbb{Q}}(|(f-f_h)(t)|,x) \leq \left\|M_n^{\mathbb{Q}}((f-f_h)(t),x)\right\|_{C_B} \leq \|f-f_h\|_{C_B} \leq \omega_2(f,h)$$

$M_n^{\mathbb{Q}}$ being a linear positive operator, gives Taylor's theorem expansion:

$$\left|M_n^{\mathbb{Q}}(f_h(t) - f_h(x),x)\right| \leq |f_h'(x)| M_n^{\mathbb{Q}}(t-x,x) + |f_h''(x)| M_n^{\mathbb{Q}}((t-x)^2,x)$$

$$\Longrightarrow \quad \left|M_n^{\mathbb{Q}}(f_h(t) - f_h(x),x)\right| \leq \|f_h'(x)\|_{C_B} M_n^{\mathbb{Q}}(t-x,x) + \|f_h''\|_{C_B} M_n^{\mathbb{Q}}((t-x)^2,x)$$

By Lemma 1 and relations (6.3) & (6.4), we get:

$$\left|M_n^{\mathbb{Q}}(f_h - f_h(x),x)\right| \leq \frac{5}{h}\omega_1(f,h)\left\{\frac{\mathbb{Q}^{n-2} + [2]_{\mathbb{Q}}}{\mathbb{Q}^n[n-2]_{\mathbb{Q}}} + \left(\frac{1}{\mathbb{Q}^{n-2}} - 1\right)x\right\} + \frac{9}{2h^2}\omega_2(f,h) M_n^{\mathbb{Q}}((t-x)^2,x)$$

where $f_h \equiv f_h(t)$ and

$$M_n^{\mathbb{Q}}((t-x)^2,x) = M_n^{\mathbb{Q}}(t^2,x) - 2xM_n^{\mathbb{Q}}(t,x) + x^2 M_n^{\mathbb{Q}}(1,x)$$

$$= \frac{[2]_{\mathbb{Q}}}{[n-2]_{\mathbb{Q}}[n-3]_{\mathbb{Q}}} + \left[\frac{\{\mathbb{Q}^5(1+2\mathbb{Q})+1\}[3]_{\mathbb{Q}}}{\mathbb{Q}^6[n-2]_{\mathbb{Q}}[n-3]_{\mathbb{Q}}} + \frac{\mathbb{Q}^5(1+2\mathbb{Q})+1}{\mathbb{Q}^{n+3}[n-2]_{\mathbb{Q}}}\right]x$$

$$+\left[\frac{\mathbb{Q}^2+\mathbb{Q}+1}{\mathbb{Q}^{n+9}[n-2]_{\mathbb{Q}}} + \frac{[3]_{\mathbb{Q}}}{\mathbb{Q}^{n+10}[n-3]_{\mathbb{Q}}} + \frac{(\mathbb{Q}^{n+2}[3]_{\mathbb{Q}} + [2]_{\mathbb{Q}}[3]_{\mathbb{Q}})}{\mathbb{Q}^{n+12}[n-2]_{\mathbb{Q}}[n-3]_{\mathbb{Q}}} + \frac{1}{\mathbb{Q}^{2n+7}}\right]x^2$$

$$-2x\left[\frac{1}{\mathbb{Q}^2[n-2]_{\mathbb{Q}}} + \frac{1}{\mathbb{Q}^n}\left(\frac{[2]_{\mathbb{Q}}}{[n-2]_{\mathbb{Q}}} + \mathbb{Q}^2\right)x\right] + x^2$$

$$= \frac{[2]_{\mathbb{Q}}}{[n-2]_{\mathbb{Q}}[n-3]_{\mathbb{Q}}} + \left[\frac{\{\mathbb{Q}^5(1+2\mathbb{Q})+1\}[3]_{\mathbb{Q}}}{\mathbb{Q}^6[n-2]_{\mathbb{Q}}[n-3]_{\mathbb{Q}}} + \frac{\mathbb{Q}^5(1+2\mathbb{Q})+1}{\mathbb{Q}^{n+3}[n-2]_{\mathbb{Q}}} - \frac{2}{\mathbb{Q}^2[n-2]_{\mathbb{Q}}}\right]x$$

$$+\left[\begin{array}{c} \dfrac{\mathbb{Q}^2+\mathbb{Q}+1}{\mathbb{Q}^{n+9}[n-2]_{\mathbb{Q}}} + \dfrac{[3]_{\mathbb{Q}}}{\mathbb{Q}^{n+10}[n-3]_{\mathbb{Q}}} + \dfrac{(\mathbb{Q}^{n+2}[3]_{\mathbb{Q}} + [2]_{\mathbb{Q}}[3]_{\mathbb{Q}})}{\mathbb{Q}^{n+12}[n-2]_{\mathbb{Q}}[n-3]_{\mathbb{Q}}} \\ + \dfrac{1}{\mathbb{Q}^{2n+7}} - \dfrac{2}{\mathbb{Q}^n}\left(\dfrac{[2]_{\mathbb{Q}}}{[n-2]_{\mathbb{Q}}} + \mathbb{Q}^2\right) + 1 \end{array}\right]x^2.$$

Setting $h = \frac{1}{\sqrt{[n-2]_{\mathbb{Q}}}} > 0$ for $x > 0$, we get the desired result.

Conclusion

By this paper we have introduced a new type of analogue of linear positive operators. This new study will give a new direction in the study of summation integral type operators in approximation theory. These operators also can be used for several type of statistical distribution functions and other functions such as Szasz, Baskakov, Beta, Gamma, and several exponential functions. Furthermore, interesting results can be obtained by new researchers in different areas.

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