IJNRD.ORG ISSN : 2456-4184
INTERNATIONAL JOURNAL OF NOVEL RESEARCH AND DEVELOPMENT (IJNRD) | IJNRD.ORG
An International Open Access, Peer-reviewed, Refereed Journal

Supersymmetry in Binomial Model Option Pricing

Authored by Sparsh Garg

Dubai, United Arab Emirates

Abstract:

Binomial model is a very prominent way of pricing options and very malleable in determining the relationships amongst prices at different nodes in the binomial tree. Supersymmetry is another well recognized and controversial theory. The combination of both these concepts can lead to a more efficient way of pricing options. Supersymmetry has the potential to illuminate insights into the financial symphony. The research talks about how the binomial model can be converted into Hamiltonian, and then the supersymmetrization of the Hamiltonian which aids in recognizing the intricate patterns in the option market in the binomial tree. It talks about entire process of incorporating supersymmetry in the binomial model and conveys the promise and power of the supersymmetry.

Introduction:

Supersymmetry is a widely acclaimed and recognized theory in modern and quantum physics. Founded by Bruno Zumino, together with Julius Wess, the theory predicts a partner for each particle in the 'Standard Model'. Supersymmetry has wide application in the field of modern finance. It's most importantly used in option pricing and market analysis. Binomial model option pricing is one of the most common methods of option pricing for American and European options. It is a very flexible model that can be easily adjusted, thereby catering to a large variety of options. Supersymmetry's implication in the binomial model option pricing is a unique concept which has not gained much attention and is not a very researched field. However, the efficiency that the advent of the beautiful mathematics behind the theory of supersymmetry can bring in pricing the option through the binomial model is unimaginable. It makes the entire system more reliable and systematized. Supersymmetry's incorporation has been heavily studied in other option pricing techniques. These involve the Black-Scholes model and the Monte Carlo simulation. The binomial model which is less popular in comparison to the other two shows strong effectiveness for the inclusion of the theory and might even yield better results than its counterpart. The binomial model alone can have several limitations. Its discrete nature might not be able to account for the continuous price movements in the market. Moreover, the model is quite computationally intensive. This might be an acute problem for the options that have large number of periods. Hence making them less suitable for complex options. However, the transformative power that the insertion of supersymmetry brings might allow the binomial model to overcome its limitations and be even more methodical compared to the other models. There are certain modifications done before supersymmetry can be implemented into this model. The system undergoes a conversion from its real value form into a representation of difference equations. The supersymmetry modules later act on these difference equations. The introduction of the theory's mathematical aspects might be necessary for the further development of the model in accordance with the current advent of technological innovations in the field of finance.

Reformation:

Converting the Binomial Model into difference equations:

The Binomial Model of pricing options is a very unique method. The model provides a generalised method for valuing an option. It is based on the principle that the current value of a stock would change its value either, up or down. When you are constructing a binomial model, you need to know the following:

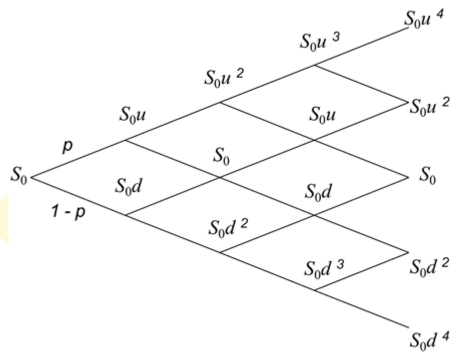
1. Beginning value of the asset (option)
2. Size of up move
3. Size of down move
4. Probability of the up move
5. Probability of the down move

Now when we are using this binomial model in the pricing of options we need to do the following steps:

Step 1 : After getting the size of the up and down value, you need to calculate the option payoff value.

Step 2 : Once you are done with this, you calculate the expected option value by multiplying the payoff values with their respective probabilities.

Step 3 : This is the last step and it involves discounting the expected value back with the help of the risk free rate.



Here's an image that showcases the binomial model for a time period of 4 years.

The first step in using the concept of supersymmetry in this model to price options will be to convert this binomial model tree into difference equations. We need to make sure that we visualise the binomial tree of the option efficiently such that there are no discrepancies in the data.

In the binomial tree each node signifies the possible price of the option at a given point of time. Initially, label the root node with its respective price as S_0 . Now label each node with its respective option price as S. Now the next step involves indicating the volume of the 'UP' movement or factor of the rise in the price (u) with its respective probability 'p', and the volume of the 'DOWN' movement or factor of the fall in the price (d) with its respective probability 'q'.

After we are done assigning the values, the next step would involve defining certain introductory variables.

Consider the following list of variables:

Price of stock at a given node in the binomial tree : S_t, i (t = time step; i = node index)

Price of the option at a given node index : V_t, i

Risk free interest rate : r

Time interval between steps : Δt

The option price are of 2 types, call option and put option. The call option at its intrinsic value or after the payoff would be equal to $S_t - k$ and the put option at its intrinsic value or after the payoff would be equal to $k - S_t$. Here k is the strike price. This holds true at the final step of calculating the difference equation.

For the earlier steps in the binomial tree, we can express the price of the option at each node as weighted average of its potential values in the next time step. The equation that would express the option price will be our difference equation.

$$V_t, i = e^{-r\Delta t} \times [p \times (V_{t+1}, i+1) + q \times (V_{t+1}, i-1)]$$

Let's look at an example for the 1st step,

At time step 0, S_0

At time step 1, S_0u or S_0d

If the node is S_0d, the difference equation will be:

$$V_1, 1 = e^{-r\Delta t} \times [p \times V_2, 2 + q \times V_2, 0]$$

Similarly, express all the nodes in the form of difference equations which would represent the option prices over different time steps and potential stock price movements.

Now, we can use this system to numerically solve all the nodes using different backward induction and matrix techniques to obtain option prices at all nodes, which will also include the initial node.

These difference equations are the core of the binomial model and the first main step in inculcating the idea of supersymmetry in binomial model option pricing. If the modelling variations are more complex, the difference equations would have additional variables or adjustments however the main logic and idea behind them would remain the same.

Introduction of the Hamiltonian Operator:

The option prices that are represented in the form of a single value at every single node in the binomial tree are converted into a 2-component state vector. The vector looks like the following:

$$| B(t, i)|$$

$$| F(t, i)|$$

$| B(t, i)|$ represents the bosonic level which comprises of the usual option price which is present at a certain node (t, i).

$| F(t, i)|$ represents the fermionic level which is an auxiliary variable that has been introduced for supersymmetry, that is initially set to 0 or 1.

Now, we will define transitioning matrices. We create 2 matrices namely M_{up} and M_{down}. These corelate with the relative up and down movements of the option's price in the binomial tree. The M_{up} transition matrix represents the state vector's transition when the option price moves up in the binomial tree. It's a 2x2 matrix that represents the fermionic and bosonic change during the upward transition. Similarly, the M_{down} matric represents the fermionic and bosonic change during the downward transition. It's also a 2x2 matrix. These 2 transition matrixes can be defined as follows:

$$M_{up} = \begin{matrix} a11 & a12 \\ a21 & a22 \end{matrix}$$

$$M_{down} = \begin{matrix} b11 & b12 \\ b21 & b22 \end{matrix}$$

Both these matrices capture the fermionic and bosonic rise and incorporates the supercharges that are involved in supersymmetry. The Hamiltonian operator kicks in after this which regulates and acts on the state vector and governs its evolution and growth over a period. The Hamiltonian operator is non- Hermitian. This is due to the directional possibilities.

The Hamiltonian operator's function can be defined as:

$$H = p \times M_{up} + q \times M_{down}$$

This operator showcases the price dynamics in the binomial model tree.

In summary, the difference equations tell us the option price at each node in the binomial model and are represented in the form of a function in accordance with its potential values in the next time step in the model. The Hamiltonian operator, thereafter, acts on the state vector condensing all these relations into a single united matrix equation.

Supersymmetry:

The operator H is non-Hermitian in nature due to the directional probabilities $(H^{\wedge} \neq H)$. It is not perfectly symmetrical, and its asymmetry leads to intricacies in understanding deeper relations of the option prices at each node in the binomial model tree. The concept of supersymmetry helps in this situation. We can directly use the theory of supersymmetry in pricing options more efficiently.

Supersymmetry would help us convert the non-Hermitian operator H into two Hermitian components. This is done so using the supercharges $(Q^{*}, Q^{\wedge})$ and specific Grassmann variables. The two components that we get include are $(Q^{*}, Q^{\wedge})$ and E (the constant energy term). We can represent this function through the following equation:

$$H = Q^{*} \times Q^{\wedge} + E$$

Q^{*} and $Q^{\wedge}$ respond onto the same state vector and change the Bosonic and Fermionic levels by 1 unit, and simultaneously adds or removes a fermionic level. Thereby, this helps in bring out the different connections and relationships between options with the same bosonic level but different fermionic level.

The conversion of the non-Hermitian matrix operator into a Hermitian operator is very important because it provides for efficient computation like diagonalization which help in making option pricing calculation less intricate.

Formation of another Isospectral Partner:

We can see from the previous equations how supersymmetry helps breaking down a non-Hermitian matrix into two Hermitian components. Now, we can construct an isospectral partner of this operator, a Hamiltonian (H') which has the same spectrum of possible option prices as the original Hamiltonian (H), but it would be having a different structure.

This Hamiltonian can be achieved through different ways. One of the ways is transformations of the supercharges.

It is represented by the following equation:

$$H' = H - i \times K$$

K is a new operator that is defined as $K = i \times (Q^{*} - Q^{\wedge})$.

The following transformation makes sure that both the Hamiltonians have the same possible option prices. This is specifically because of their connection with K and the mathematical properties of the supercharges.

The H' is very useful in financial modelling as it provides for a fresh perspective as it gives a more modern lens to analyze the option pricing problems and give a more efficient way of predicting in the binomial model tree. H' is mathematically more conveniently traceable as compared to the original H operator. The derived relationship from the two Hamiltonians can help provide for better option pricing strategies and risk management methods.

Using the Pricing Kernel Computation (Stochastic discount factor):

The usual approach to this method would have involved calculating the risk neutral probabilities from the original Hamiltonian H, and then iterating backwards in the binomial tree which would also involve the discounting of the future payoffs with risk neutral probabilities to obtain option prices. However, using supersymmetry would change the way we use the stochastic discount factor. There is ongoing research on price kernel computation in the field of supersymmetry. There hasn't been a well-defined fully functional way of using the price kernel computation through a supersymmetrical approach. However, a few hypothesizes might suggest expressing the price kernel in terms of the supercharges and then find its relationship with the two Hamiltonians.

References:

Jana, T. K., & Roy, P. K. (2011b). Supersymmetry in option pricing. *Physica A: Statistical Mechanics and Its Applications*, *390*(12), 2350–2355. https://doi.org/10.1016/j.physa.2011.02.027

One, A. (2023, July 12). *Understanding the binomial option pricing model*. Angel One. https://www.angelone.in/knowledge-center/futures-and-options/binomial-option-pricing-model#:~:text=The%20binomial%20option%20pricing%20model%20is%20a%20discrete%2Dtime%20model,for%20options%20with%20long%20maturities.

Menchions, Y. (2023, July 31). *Exactly solvable diffusion equations and pricing models based on exceptional hermite polynomials*. https://dalspace.library.dal.ca/handle/10222/82743

Vahdati, S., Fardi, M., & Ghasemi, M. (2018). Option pricing using a computational method based on reproducing kernel. *Journal of Computational and Applied Mathematics*, *328*, 252–266. https://doi.org/10.1016/j.cam.2017.05.032

Cotfas, L. (2013, October 15). *Quantum harmonic oscillator in option pricing*. arXiv.org. https://arxiv.org/abs/1310.4142

Team, W. (2023, January 23). *Binomial Option Pricing Model*. WallStreetMojo. https://www.wallstreetmojo.com/binomial-option-pricing-model/

Basic Binomial options pricing model. (n.d.). Stack Overflow. https://stackoverflow.com/questions/59382046/basic-binomial-options-pricing-model

Bard, Google AI. "How can supersymmetry be used in Binomial model for option pricing" Bard, Google AI, [2023-01-04]. Accessed [2023-01-04].