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"An explanation of Transport and Heat equation by 2 and n-dimensional Laplace Transform"

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1. Abstract:

There is lots of research available in 1 or 2 dimensional Laplace transform but little research available for n - Dimensional Laplace transform, in this paper we develop a table for n-dimensional Laplace transform and make a relation of heat equation $u(x, t) = e^{-\frac{1}{\lambda}t}$ and transport equation $\boldsymbol{u}(\boldsymbol{x}, \boldsymbol{y}) = \left[\boldsymbol{f}(\boldsymbol{x} - \frac{\boldsymbol{a}}{\boldsymbol{b}}\boldsymbol{y})\right]$.

2. Kew Words: Laplace Transform, Transport equation, Heat equation, Partial differential equation etc.

3. Introduction:

Integral transform has effective tools to solve many science and engineering problems. Laplace transform is one of them important integral to solve the same problems. This has effectively been used in finding the solutions of linear ordinary and partial differential, difference and integral equations.

On the other hand, Joseph Fourier's (1768–1830) monumental treatise on La Theories[6], [8]´.Analytic de la Chaleur (The Analytical Theory of Heat) provided the modern mathematical theory of heat conduction, Fourier series, and Fourier integrals with applications. In this series of application we can also find solution of transport equation by using double lateral Laplace transform.

The double Laplace transform of a function *f (x, y)* of two variables *x* and *y* defined in the first quadrant of the x–y plane is defined by the double integral in the form

$$L_2[f(x,y);s_1,s_2] = \int_0^\infty \int_0^\infty e^{-s_1x-s_2y} f(x,y)dxdy = u(s_1,s_2) \quad \ldots(1)$$

Where

$$u(s_1,s_2) = L_2[f(x,y)] = L[L\{f(x,y): x \to s_1\}: y \to q] = L\{u(s_1,y): y \to q$$

Provided the integral exists, where we fallow Debnath and Bhatta [4] to denote Laplace transform $\overline{f(s)}$ = L{f(x)}= $\int_0^\infty e^{-sx} f(x)dx$, Re(s)>0 ...(2)

The inverse double Laplace Transform $L_2^{-1}\{ u(s_1,s_2)\}$= f(x, y) is defined by the complex double integral

$$L_2^{-1}\{ u(s_1,s_2)\} = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} e^{s_1x} ds_1 \frac{1}{2\pi i}\int_{d-i\infty}^{d+i\infty} e^{s_2x} u(s_1,s_2)ds_2 \quad \ldots(3)$$

Where $u(s_1,s_2)$must be an analytic function for all s_1 and s_2 in the region defined by the inequalities Re(s_1)>c and Re(s_2)>d, where c and d are arbitrary suitable constant.

3.1 Linear Property of double Laplace and Inverse Laplace Transform:

1. **Laplace transform**

$$L_2\{a_1f_1(x,y) + a_2f_2(x,y)\} = \int_0^\infty \int_0^\infty e^{-s_1x-s_2y}\{a_1f_1(x,y) + a_2f_2(x,y)\}dxdy$$

$$=a_1 \int_0^\infty \int_0^\infty e^{-s_1x-y} f_1(x,y)dxdy + a_2 \int_0^\infty \int_0^\infty e^{-s_1x-s_2y} f_2(x,y)\}dxdy$$

$=a_1L_2\{f_1(x,y)\} + a_2L_2\{f_2(x,y)\}$, where a_1 and a_2 are constant.

2. **Inverse Laplace Transform:**

$$L_2^{-1}\{a_1u_1(s_1,s_2) + a_2u_2(s_1,s_2)\} = a_1 L_2^{-1}\{u_1(s_1,s_2)\} + a_2L_2^{-1}\{u_2(s_1,s_2)\}$$

3.2 Examples:

1. If f(x, y) = 1 for x>0, y>0,then

$$L_2\{1\} = \int_0^\infty \int_0^\infty e^{-s_1x-s_2y}.1dxdy$$

$$= \int_0^\infty e^{-s_1x}dx. \int_0^\infty e^{-s_2y}.1dxdy$$

$$= \frac{1}{s_1}\frac{1}{s_2}$$

$$L_2\{1\} = \frac{1}{s_1s_2}$$

2. If f(x, y)= exp(ax+by) for all x and y, then

$$L_2\{exp(ax+by)\} = \int_0^\infty \int_0^\infty e^{-s_1x-s_2y}.e^{ax+by}dxdy$$

$$= \int_0^\infty e^{-(s_1-a)x} dx \int_0^\infty e^{-(s_2-b)y}dy$$

$$= \frac{1}{(s_1-a)(s_2-b)} \text{for} s_1 > a, s_2 > b$$

3. If f(x, y)= exp[i(ax+by)] for all x and y, then

$$L_2\{exp\ i(ax+by)\} = \int_0^\infty \int_0^\infty e^{-s_1x-s_2y}.e^{i(ax+by)}dxdy$$

$$= \int_0^\infty e^{-(s_1-ai)x}\,dx \int_0^\infty e^{-(s_2-bi)y}dy$$

$$= \frac{1}{(s_1-ai)(s_2-bi)} \text{ or } \frac{(s_1s_2-ab)+i(as_1+bs_2)}{(s_1^2+a^2)(s_2^2+b^2)}$$

4. If f(x, y) = cos(ax+ by) for all x and y

Then

$$L_2\{\cos(ax+by)\} = \int_0^\infty \int_0^\infty e^{-s_1x-s_2y}.\cos(ax+by)\,dxdy$$

$$L_2\{\cos(ax+by)\} = \int_0^\infty \int_0^\infty e^{-s_1x-s_2y}.\frac{e^{i(ax+by)}+e^{-i(ax+by)}}{2}dxdy$$

$$L_2\{\cos(ax+b) = \frac{1}{2}\int_0^\infty \int_0^\infty e^{-(s_1-ai)x}e^{-(s_1-ai)x}dxdy + \frac{1}{2}\int_0^\infty \int_0^\infty e^{-(s_1+ai)x}e^{-(s_1+ai)x}dxdy$$

$$= \frac{1}{2}\left[\frac{1}{(s_1-ai)}\frac{1}{(s_2-bi)} + \frac{1}{(s_1+ai)}\frac{1}{(s_2+bi)}\right] \text{ or}$$

$$L_2\{\cos(ax+by)\} = \frac{1}{2}\left[\frac{s_1s_2+s_1bi+s_2ai-ab+s_1s_2-s_1bi-s_2ai-ab}{(s_1^2+a^2)(s_2^2+b^2)}\right]$$

$$L_2\{\cos(ax+by)\} = \left[\frac{s_1s_2-ab}{(s_1^2+a^2)(s_2^2+b^2)}\right]$$

Similarly we can find all sin(ax+by), sinh(ax+by), cosh(ax+by)
Now

$$L_2\{\sin(ax+by)\} = \left[\frac{as_2+bs_1}{(s_1^2+a^2)(s_2^2+b^2)}\right]$$

$$L_2\{\cosh(ax+by)\} = \left[\frac{s_1s_2-ab}{(s_1^2-a^2)(s_2^2-b^2)}\right]$$

$$L_2\{\sinh(ax+by)\} = \left[\frac{as_2+bs_1}{(s_1^2-a^2)(s_2^2-b^2)}\right]$$

3.3 Table for Double Laplace Transform:

f(x, y)	$L_2[f(x,y);s_1,s_2]$
1	$\frac{1}{s_1s_2}$
x^my^n	$\frac{m!.n!}{s_1^{m+1}s_2^{n+1}}$ or $\frac{\Gamma(m+1)\Gamma(n+1)}{s_1^{m+1}s_2^{n+1}}$
$e^{(ax+by)}$	$\frac{1}{(s_1-a)(s_2-b)}$ for $s_1 > a, s_2 > b$
$e^{i(ax+by)}$	$\frac{1}{(s_1-ai)(s_2-bi)}$ or $\frac{(s_1s_2-ab)+i(as_1+bs_2)}{(s_1^2+a^2)(s_2^2+b^2)}$

cos(ax+by)	$\left[\frac{s_1 s_2 - ab}{(s_1^2 + a^2)(s_2^2 + b^2)}\right]$
sin(ax+by)	$\left[\frac{a s_2 + b s_1}{(s_1^2 + a^2)(s_2^2 + b^2)}\right]$
cosh(ax+by)	$\left[\frac{s_1 s_2 - ab}{(s_1^2 - a^2)(s_2^2 - b^2)}\right]$
sinh(ax+by)	$\left[\frac{a s_2 + b s_1}{(s_1^2 - a^2)(s_2^2 - b^2)}\right]$

3.4 Existence Condition for the double Laplace Transform:

If f(x, y) is said to be of *exponential order* a (>0) and b(>0) on $0 \le x < \infty$, $0 \le y < \infty$, if there exists a positive constant K such that for all x >0 and y >0

$|f(x, y)| \le K e^{ax+by}$ and we write $f(x, y) = O(e^{ax+by})$ as $x \to \infty, y \to \infty$.

Or, equivalently,

$\lim_{x\to\infty, y\to\infty} e^{-\alpha x - \beta y}|f(x,y)| = K.$ $\lim_{x\to\infty, y\to\infty} e^{-(\alpha-a)x-(\beta-b)y} = 0$, $\alpha > a$, $\beta > b$

Such a function f(x,y) is simply called an *exponential* order as $x \to \infty$, $y \to \infty$, and clearly, it does not grow faster than K exp(ax + by) as $x \to \infty$, $y \to \infty$.

3.5 Existence Theorem: If a function f(x, y) is a continuous function in every finite intervals (0,x) and (0,y) and of exponential order exp(ax + by), then the double Laplace transform of f(x,y) exists for all s_1 and s_2 provided Re $s_1 > a$ and Re $s_2 > b$.

Proof We have

$$|u(s_1, s_2)| = \left|\int_0^\infty \int_0^\infty e^{-s_1 x - s_2 y} f(x,y)\,dx\,dy\right|$$

$$\le k \int_0^\infty e^{-(s_1-a)x} dx \int_0^\infty e^{-(s_2-b)y} dy$$

$$= \frac{1}{(s_1-a)(s_2-b)} \text{ for } Re(s_1)>0, Re(s_2)>0 \quad (3.7.1)$$

Therefore

$$\lim_{s_1\to\infty, s_2\to\infty} |u(s_1,s_2)| = 0 \text{ or } \lim_{s_1\to\infty, s_2\to\infty} u(s_1,s_2) = 0$$

This result can be regarded as the limiting property of the double Laplace transform. Clearly, $u(s_1, s_2) = s_1 . s_2$ **or** $s_1^2 + s_2^2$ is not the double Laplace transform of any function f(x,y) because $u(s_1, s_2)$ does not tend to zero as $s_1 \to \infty$ and $s_2 \to \infty$.

On the other hand, $f(x,y) = \exp(ax^2 + by^2)$, a >0, b >0 cannot have a double Laplace transform even though it is continuous but is *not* of the exponential order because

Lim $\{\exp(ax^2+by^2-s_1x-s_2y)\} = \infty$ as $x\to\infty, y\to\infty$

3.6 Basic Property of the double Laplace Transform:

By using Debnath and Bhatta,[4] we can prove the following general properties of the double Laplace transform under suitable condition on f(x, y):

a. $L_2\left[e^{(-ax-by)} f(x,y)\right] = u(s_1 + a, s_2 + b)$

b. $L_2[f(ax)g(by)] = \frac{1}{ab} u\left(\frac{s_1}{a}\right) u\left(\frac{s_2}{b}\right) \; a > 0, b > 0$

c. $L_2[f(x)] = \frac{1}{s_2}u(s_1), \quad L_2[g(y)] = \frac{1}{s_1}u(s_2),$

d. $L_2[f(x+y)] = \frac{1}{s_1-s_2}[u(s_1) - u(s_2)]$

e. $L_2[f(x-y)] = \frac{1}{s_1+s_2}[u(s_1) + u(s_2)]$, **when f is even**

$= \frac{1}{s_1+s_2}[u(s_1) - u(s_2)]$, **when f is odd**

f. $L_2[f(x)H(x-y)] = \frac{1}{s_2}[u(s_1) - u(s_1+s_2)]$

g. $L_2[f(x)H(y-x)] = \frac{1}{s_2}[u(s_1+s_2)]$

h. $L_2[f(x)H(y+x)] = \frac{1}{s_2}[u(s_1)]$

i. $L_2[H(x-y)] = \frac{1}{s_1(s_1+s_2)}, put\ f(x) = 1$

j. $L_2\left[\frac{\partial u}{\partial x}\right] = s_1\, u(s_1,s_2) - u_1(s_2)$,**where &**$u_1(s_2)$**= L{u(0, y)}**

k. $L_2\left[\frac{\partial u}{\partial y}\right] = s_2\, u(s_1,s_2) - u_2(s_1)$,**where &**$u_2(s_1)$**= L{u(x, 0)}**

l. $L_2\left[\frac{\partial^2 u}{\partial x^2}\right] = {s_1}^2\, u(s_1,s_2) - s_1u_1(s_2) - u_3(s_2) where\ u_3(s_2) = Lu_x(0,y)$

m. $L_2\left[\frac{\partial^2 u}{\partial y^2}\right] = {s_2}^2\, u(s_1,s_2) - s_2u_2(s_1) - u_3(s_1) where\ u_4(s_1) = Lu_y(x,0)$

n. $L_2\left[\frac{\partial^2 u}{\partial x \partial y}\right] = s_1s_2\, u(s_1,s_2) - s_2u_1(s_2) - s_1u_2(s_2) + u(0,0) where\ Lu_x(x,0) = s_1u_2(s_1) - u(0,0)$

4. **Solution of transport equation by Double Laplace Transforms:**

We find solution of the transport equation $au_x + bu_y = 0$ with the condition u(x,0)= f(x), x>0; u(0,y) = 0 , y>0

Solution:

Given $au_x + bu_y = 0$ with initial condition u(x,0)= f(x), x>0; u(0,y) = 0 , y>0

Taking Laplace both side

$$aL_2\{u_x(x,y)\} + bL_2\{u_y(x,y)\} = 0$$

$$a[s_1\, u(s_1,s_2) - L\{u(0,y)\}] + b[s_2\, u(s_1,s_2) - L\{u(x,0)\}] = 0$$

$$a[s_1\, u(s_1,s_2) - 0] + b[s_2\, u(s_1,s_2) - f(s_1)] = 0$$

$$(as_1 + bs_2)\, u(s_1,s_2) = bf(s_1)$$

$$u(s_1,s_2) = \frac{bf(s_1)}{(as_1+bs_2)}$$

$$u(s_1,s_2) = \frac{f(s_1)}{\left(\frac{a}{b}s_1+s_2\right)}$$

Taking Laplace inverse with respect to s_2

$$u(s_1,y) = f(s_1)exp\left(-\frac{a}{b}s_1y\right)$$

Now again taking inverse Laplace with respect to s_1

$$u(x,y) = L_2^{-1}\left\{f(s_1)exp\left(-\frac{a}{b}s_1y\right)\right\}$$

By first shifting property of double Laplace Transform

$$\boldsymbol{u}(\boldsymbol{x},\boldsymbol{y}) = \left[\boldsymbol{f}\left(\boldsymbol{x}-\frac{\boldsymbol{a}}{\boldsymbol{b}}\boldsymbol{y}\right)\right] \dots (4.1)$$

Special case:

For a = 1 and b = -1, then $u_x = u_y$ with the condition u(x,0)= f(x), x>0; u(0,y) = g(y) , y>0

Taking Laplace both side,

$$[s_1\, u(s_1,s_2) - L\{u(0,y)\}] = [s_2\, u(s_1,s_2) - L\{u(x,0)\}]$$

$$[s_1\, u(s_1,s_2) - g(s_2)] = [s_2\, u(s_1,s_2) - f(s_1)]$$

$$(s_1 - s_2)u(s_1,s_2) = g(s_2) - f(s_1)$$

$$u(s_1,s_2) = \frac{g(s_2)-f(s_1)}{(s_1-s_2)}$$

Taking inverse Laplace transform we get

$$u(x,y) = L_2^{-1}\left\{\frac{g(s_2)-f(s_1)}{(s_1-s_2)}\right\} \dots\dots (4.2)$$

In particular case if u(x,0) = 1 & u(0, y) = 1 then $\boldsymbol{L\{u(x,0)\}} = \frac{1}{s_1}$ & $\boldsymbol{L\{u(0,y)\}} = \frac{1}{s_2}$ **,** then from equation

$$u(x,y) = L_2^{-1}\left\{\frac{\frac{1}{s_2}-\frac{1}{s_1}}{(s_1-s_2)}\right\}$$

$$u(x,y) = L_2^{-1}\left\{\frac{1}{(s_1.s_2)}\right\}$$

u(x, y) = 1 …(4.3)

5. Partial Differential Equation for heat conduction: A typical representative of parabolic PDE is *heat conduction*. A model of heat conduction in a rod equipped with an array of temperature sensors and heaters is schematically sketched in a rod of length x. It is described by well-known heat equation. [11]

$\frac{\partial u(x,t)}{\partial t} = k\frac{\partial^2 u(x,t)}{\partial x^2} + f(x,t) \dots$ (5.1) with initial condition u(0, t)= 1, u_x(0, t)} = 0 and u(x,0) = 1, for k = 1

Where *u* denotes temperature (^{0}C), *f* the input heat (^{0}C s^{-1}), *t* and *x* denote time (s) and a spatial coordinate (m), respectively, and $\kappa = \frac{\chi}{\rho c_p}$, is a constant ($m^2$ s^{-1}), where κ is the thermal conductivity (W $m^{-1}k^{-1}$) , *ρ* is the density(kgm^{-3}) and c_p is the heat capacity per unit mass (J K^{-1} kg^{-1}).

Since when we give input heat in to the rod then temperature is directly proportional to input heat. So $u(x,t) \propto f(x,t)$, then we can say $u(x,t) = \lambda f(x,t)$, where λ is heat resistive coefficient. Then $f(x,t) = \frac{1}{\lambda}u(x,t)$ … (5.2) with initial if u(0, t)=1, then f(0, t) = $\frac{1}{\lambda}$, if $u(x,0) = 1$, then $f(x,0) = \frac{1}{\lambda}$ and f_x(0, t) = 0.

Taking Laplace both side in equation (5.1)

$$L_2\{u_t(x,t)\} = kL_2\{u_{xx}(x,t)\} + L_2\{f(x,t)\}$$

$$S_2\check{\check{u}}(s_1,s_2) - L\{u(x,0)\} = k\left[s_1^2\check{\check{u}}(s_1,s_2) - s_1L\{u(0,t) - L\{u_x(0,t)\}\right] + L_2\left\{\frac{1}{\lambda}u(x,t)\right\} \dots (5.3)$$

$$s_2\breve{\breve{u}}(s_1,s_2) - \frac{1}{s_1} = k\left[s_1^2\breve{\breve{u}}(s_1,s_2) - s_1\frac{1}{s_2}\right] + \left\{\frac{1}{\lambda}u(s_1,s_2)\right\}$$

$$\{s_2 - ks_1^2\}\breve{\breve{u}}(s_1,s_2) = k\frac{s_1}{s_2} + \frac{1}{s_1} + \left\{\frac{1}{\lambda}u(s_1,s_2)\right\}$$

$$\{s_2 - ks_1^2 - \tfrac{1}{\lambda}\}\breve{\breve{u}}(s_1,s_2) = k\frac{s_1}{s_2} + \frac{1}{s_1}$$

$$\breve{\breve{u}}(s_1,s_2) = k\frac{s_1}{s_2\{s_2 - ks_1^2 - \frac{1}{\lambda}\}} + \frac{1}{s_1\{s_2 - ks_1^2 - \frac{1}{\lambda}\}}$$

$$\breve{\breve{u}}(s_1,s_2) = k\frac{s_1}{s_2\{s_2 - ks_1^2 - \frac{1}{\lambda}\}} + \frac{1}{s_1\{s_2 - ks_1^2 - \frac{1}{\lambda}\}}$$

If k = 1 then above equation can be written as

$$\breve{\breve{u}}(s_1,s_2) = \frac{s_1}{s_2\{s_2 - s_1^2 - \frac{1}{\lambda}\}} + \frac{1}{s_1\{s_2 - s_1^2 - \frac{1}{\lambda}\}}$$

$$\breve{\breve{u}}(s_1,s_2) = -\frac{s_1}{s_2\left\{s_1^2 - s_2 + \frac{1}{\lambda}\right\}} - \frac{1}{s_1\{s_1^2 - s_2 + \frac{1}{\lambda}\}}$$, by using partial fraction this equation can be written as

$$\breve{\breve{u}}(s_1,s_2) = \frac{s_1}{(s_1^2 - \frac{1}{\lambda})}\left[\frac{1}{\{s_2 - (s_1^2 - \frac{1}{\lambda})\}} - \frac{1}{s_2}\right] + \frac{1}{s_1\{s_2 - (s_1^2 - \frac{1}{\lambda})\}}$$

Taking inverse Laplace with respect to s_2

$$\breve{u}(s_1,t) = \frac{s_1}{(s_1^2 - \frac{1}{\lambda})}\left[e^{\left(s_1^2 - \frac{1}{\lambda}\right)t} - 1\right] + \frac{1}{s_1}e^{\left(s_1^2 - \frac{1}{\lambda}\right)t}$$

$$\breve{u}(s_1,t) = \frac{s_1}{(s_1^2 - \frac{1}{\lambda})}.e^{\left(s_1^2 - \frac{1}{\lambda}\right)t} - \frac{s_1}{(s_1^2 - \frac{1}{\lambda})} + \frac{1}{s_1}e^{\left(s_1^2 - \frac{1}{\lambda}\right)t}$$

$$since\ e^{\left(s_1^2 - \frac{1}{\lambda}\right)t} = 1 + \left(s_1^2 - \frac{1}{\lambda}\right)t + \frac{\{\left(s_1^2 - \frac{1}{\lambda}\right)t\}^2}{2!} + \frac{\{\left(s_1^2 - \frac{1}{\lambda}\right)t\}^3}{3!} + \cdots$$

Then above equation can be written as:

$$\breve{u}(s_1,t) = \frac{s_1}{\left(s_1^2 - \frac{1}{\lambda}\right)}\left[1 + \frac{\left(s_1^2 - \frac{1}{\lambda}\right)t}{1!} + \frac{\left\{\left(s_1^2 - \frac{1}{\lambda}\right)t\right\}^2}{2!} + \frac{\left\{\left(s_1^2 - \frac{1}{\lambda}\right)t\right\}^3}{3!} + \cdots\right] - \frac{s_1}{\left(s_1^2 - \frac{1}{\lambda}\right)} + \frac{1}{s_1}\left[1 + \left(s_1^2 - \frac{1}{\lambda}\right)t + \frac{\{\left(s_1^2 - \frac{1}{\lambda}\right)t\}^2}{2!} + \frac{\{\left(s_1^2 - \frac{1}{\lambda}\right)t\}^3}{3!} + \cdots\right]$$

$$\breve{u}(s_1,t) = \left[\frac{s_1}{\left(s_1^2 - \frac{1}{\lambda}\right)} + \frac{s_1 t}{1!} + \frac{\left(s_1^2 - \frac{1}{\lambda}\right).s_1.t^2}{2!} + \frac{\left\{\left(s_1^2 - \frac{1}{\lambda}\right)\right\}^2 s_1.t^3}{3!} + \cdots\right] - \frac{s_1}{\left(s_1^2 - \frac{1}{\lambda}\right)} + [\frac{1}{s_1} + \frac{(s_1 - \frac{1}{\lambda s_1})t}{1!} + \frac{\left(s_1^3 - \frac{2}{\lambda}s_1 + \frac{1}{\lambda^{2s_1}}\right)t^2}{2!} + \cdots.]$$

Now taking inverse Laplace along S_1

$$u(x,t) = \left[cosh\sqrt{\frac{1}{\lambda}}x + 0 + 0 + \cdots\right] - cosh\sqrt{\frac{1}{\lambda}}x + 1 - \frac{1}{\lambda}t + \frac{\left(\frac{1}{\lambda}t\right)^2}{2!} - \frac{\left(\frac{1}{\lambda}t\right)^3}{3!} + \cdots$$

Therefore

$$u(x,t) = e^{-\frac{1}{\lambda}t} \quad ...(5.4)$$

Here we see that u(x, t) is free from x so we can say the decay of temperature depend only on time not the length of rod. That is when the thermal conductivity is taken as unity (k = 1), that is on unit thermal conductivity the input heat in the rod decrease its temperature with time and u(x, t) does not depend distance from one end. It is also shows that it is bounded as $t \to \infty$, the temperature $u(x,t) \to 0$.

6. N-dimensional Laplace Transform for the solution of boundary value problem:

6.1 N-dimensional Laplace Transform:

The N-dimensional Laplace transform have many applications into the field of applied science, engineering and physics etc. Let us consider a n-dimensional function f(t_1, t_2,t_3,…, t_n), then N-dimensional Laplace transform defined as $L_n\{f(t_1, t_2, t_3, \ldots t_n)\} =$

$$\int_0^\infty \int_0^\infty \int_0^\infty \ldots \int_0^\infty e^{-s_1t_1-s_2t_2\ldots-s_nt_n f(t_1,t_2,t_3,\ldots t_n)}\, dt_1dt_2dt_3 \ldots dt_n = f(s_1, s_2, s_3, \ldots s_n) \quad ,$$

where s is any parameter (Real or complex). [12]

Jaeger use multidimensional Laplace transform for boundary value problems in heat conduction [5] and earlier it was given some theoretical result on multidimensional Laplace Transform by R.S. Dahiya and Debnath Battha.[3][4][5].In this chapter we will develop n dimensional Laplace transform table and its few property with some important theorems on multidimensional Laplace transform with few important examples and put solution of partial differential equation on boundary value problems by multidimensional Laplace transform.

Note: we use x,y for two variable 2-dimensional Laplace transform, but here we are trying to find N- dimensional Laplace transform for multivariable function so we use t_1, t_2, t_3,…t_n in place of x and y

6.2 Linear Property of N-dimensional Laplace Transform:

$$L_n\{f(t_1, t_2, t_3, \ldots t_n) \pm g(t_1, t_2, t_3, \ldots t_n) = L_n\{f(t_1, t_2, t_3, \ldots t_n)\} + L_n\{g(t_1, t_2, t_3, \ldots t_n)$$

6.3 N- Dimensional Laplace Transform of some important functions:

Here we are trying to develop n-dimensional Laplace transform table:

1. We already know $L_1\{1\} = \frac{1}{s_1}$

$$L_2\{1\} = \int_0^\infty \int_0^\infty e^{-s_1t_1-s_2t_2}.1dt_1dt_2$$

$$L_2\{1\} = \int_0^\infty 1\, e^{-s_1t_1}dt_1.\int_0^\infty 1e^{-s_2t_2}dt_2$$

$$= \frac{1}{s_1}\frac{1}{s_2}$$

$$L_3\{1\} = \int_0^\infty 1\, e^{-s_1t_1}dt_1 . \int_0^\infty 1e^{-s_2t_2}dt_2 \int_0^\infty 1e^{-s_3t_3}dt_3$$

$$= \frac{1}{s_1}\frac{1}{s_2}\frac{1}{s_3}$$

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$$L_n\{1\} = \int_0^\infty 1\, e^{-s_1t_1}dt_1 . \int_0^\infty 1e^{-s_2t_2}dt_2 \int_0^\infty 1e^{-s_3t_3}dt_3 \dots \int_0^\infty 1\, e^{-s_nt_n}dt_n .$$

$$L_n\{1\} == \frac{1}{s_1}\frac{1}{s_2}\frac{1}{s_3}\dots\frac{1}{s_n}$$

2. $L_1\{t_1\}, L_2\{t_1 + t_2\}$, Already proved in earlier chapter now for N- dimensional Laplace Transform we are trying to find in 3, 4 …up to n-dimensional Laplace.
So we have

$$L_3\{t_1 + t_2\} = \int_0^\infty \int_0^\infty \int_0^\infty e^{-s_1t_1-s_2t_2-s_3t_3}.\{t_1 + t_2\}\, dt_1dt_2dt_3$$

$$= \int_0^\infty \int_0^\infty \int_0^\infty e^{-s_1t_1-s_2t_2-s_3t_3}.\{t_1\}\, dt_1dt_2dt_3 + \int_0^\infty \int_0^\infty \int_0^\infty e^{-s_1t_1-s_2t_2-s_3t_3}.\{t_2\}\, dt_1dt_2dt_3$$

$$= \int_0^\infty e^{-s_1t_1}\, t_1dt_1 . \int_0^\infty e^{-s_2t_2}\, dt_2 . \int_0^\infty e^{-s_3t_3}\, dt_3 + $$
$$+ \int_0^\infty e^{-s_1t_1}\, dt_1 . \int_0^\infty e^{-s_2t_2} t_2\, dt_2 . \int_0^\infty e^{-s_3t_3}\, dt_3$$

On integration we get:

$$= \frac{1}{s_1^2}.\frac{1}{s_2}\frac{1}{s_3} + \frac{1}{s_2^2}.\frac{1}{s_1}\frac{1}{s_3}$$

Similarly

$$L_n\{t_1 + t_2\} = \frac{1}{s_1^2}.\frac{1}{s_2}\frac{1}{s_3 \dots s_n} + \frac{1}{s_2^2}.\frac{1}{s_1}\frac{1}{s_3 \dots s_n}$$

$$L_3\{t_1 + t_2 + t_3\} = \int_0^\infty \int_0^\infty \int_0^\infty e^{-s_1t_1-s_2t_2-s_3t_3}.\{t_1 + t_2 + t_3\}\, dt_1dt_2dt_3$$

$$= \int_0^\infty \int_0^\infty \int_0^\infty e^{-s_1t_1-s_2t_2-s_3t_3}.\{t_1\}\, dt_1dt_2dt_3 + \int_0^\infty \int_0^\infty \int_0^\infty e^{-s_1t_1-s_2t_2-s_3t_3}.\{t_2\}\, dt_1dt_2dt_3 . \int_0^\infty \int_0^\infty \int_0^\infty e^{-s_1t_1-s_2t_2-s_3t_3}.\{t_3\}$$

$$= \int_0^\infty e^{-s_1t_1}\, t_1dt_1 . \int_0^\infty e^{-s_2t_2}\, dt_2 . \int_0^\infty e^{-s_3t_3}\, dt_3 +$$
$$\int_0^\infty e^{-s_1t_1}\, t_1dt_1 . \int_0^\infty e^{-s_2t_2}\, dt_2 . \int_0^\infty e^{-s_3t_3}\, dt_3$$
$$+ \int_0^\infty e^{-s_1t_1}\, t_1dt_1 . \int_0^\infty e^{-s_2t_2}\, dt_2 . \int_0^\infty e^{-s_3t_3}\, dt_3$$

On integration we get:

$$= \frac{1}{s_1^2}.\frac{1}{s_2}\frac{1}{s_3} + \frac{1}{s_2^2}.\frac{1}{s_1}\frac{1}{s_3} + \frac{1}{s_3^2}.\frac{1}{s_2}\frac{1}{s_1}$$

Similarly we can find
4[th] ,5[th] …and N-dimensionl Laplace Transform

$$L_n\{t_1+t_2+t_3\ldots+t_n\} = \int_0^\infty\int_0^\infty\int_{0\ldots n\,times}^\infty e^{-s_1t_1-s_2t_2-s_3t_3\ldots+s_nt_n}.\{t_1+t_2+t_3\ldots+t_n\}\ dt_1dt_2dt_3\ldots dt_n$$

$$=\frac{1}{s_1^2}.\frac{1}{s_2}\frac{1}{s_3\ldots s_n}+\frac{1}{s_2^2}.\frac{1}{s_1}\frac{1}{s_3\ldots s_n}+\frac{1}{s_3^2}.\frac{1}{s_2}\frac{1}{s_1\ldots s_n}+\cdots+.\frac{1}{s_1s_2s_3\ldots s_n^2}$$

3. We have

$$L_2\{(t_1+t_2)^2\} = L_2\{(t_1)^2\}+L_2\{(t_2)^2\}+2L_2\{t_1t_2\}$$

So taking

$$L_2\{(t_1)^2\} = \int_0^\infty\int_0^\infty e^{-s_1t_1-s_2t_2}.(t_1)^2dt_1dt_2$$

$$=\int_0^\infty e^{-s_1t_1}(t_1)^2dt_1\int_0^\infty e^{-s_2t_2}dt_2$$

$$=\frac{2!}{s_1^3.s_2}$$

Simalarly $L_2\{(t_2)^2\}=\frac{2!}{s_2^3s_1}$

And $L_2\{t_1.t_2\}=\frac{1}{s_1^2s_2^2}$ therefore

$$L_2\{(t_1+t_2)^2\}=\frac{2!}{s_1^3.s_2}+\frac{2!}{s_2^3s_1}+\frac{2}{s_1^2s_2^2}$$

$$=\frac{2}{s_1s_2}\left[\frac{1}{s_1^2}+\frac{1}{s_2^2}+\frac{1}{s_1s_2}\right] \text{ or } =\frac{2(s_1^2+s_2^2+s_1s_2)}{s_1^3s_2^3}$$

$$L_3\{(t_1+t_2)^2\} = L_3\{(t_1)^2\}+L_3\{(t_2)^2\}+2L_3\{t_1t_2\}$$

$$=\frac{2!}{s_1^3.s_2s_3}+\frac{2!}{s_2^3s_1s_3}+\frac{2}{s_1^2s_2^2s_3}$$

$$=\frac{2}{s_1s_2s_3}\left[\frac{1}{s_1^2}+\frac{1}{s_2^2}+\frac{1}{s_1s_2}\right] \text{ or } \frac{2!(s_1^2+s_2^2+s_1s_2)}{s_1^3s_2^3s_3}$$

Similarly we get

$$L_n\{(t_1+t_2)^2\}=\frac{2!}{s_1s_2s_3\ldots s_n}\left[\frac{1}{s_1^2}+\frac{1}{s_2^2}+\frac{1}{s_1s_2}\right] or \frac{2!(s_1^2+s_2^2+s_1s_2)}{s_1^3s_2^3s_3\ldots s_n}$$

Now we can find all N-dimensional Laplace transform of $(t_1+t_2)^3,\ldots(t_1+t_2)^n$

4. $L_2\{e^{a_1t_1+a_2t_2}\}=\int_0^\infty\int_0^\infty e^{-s_1t_1-s_2t_2}.e^{a_1t_1+a_2t_2}dt_1dt_2$

$$=\int_0^\infty e^{-s_1t_1}.e^{a_1t_1}dt_1.\int_0^\infty e^{-s_2t_2}.e^{a_2t_2}dt_2$$

$$=\int_0^\infty e^{-(s_1-a_1)t_1}dt_1.\int_0^\infty e^{-(s_2-a_2)t_2}dt_2$$

where $s_i > a_i$ for i= 1,2 then

$$L_2\{e^{a_1t_1+a_1t_1}\}=\frac{1}{(s_1-a_1)(s_2-a_2)}$$

$$Now\ L_3\{e^{a_1t_1+a_2t_2}\}=\int_0^\infty\int_0^\infty\int_0^\infty e^{-s_1t_1-s_2t_2-s_3t_3}.\{e^{a_1t_1+a_1t_1}\}\ dt_1dt_2dt_3$$

$$=\int_0^\infty e^{-(s_1-a_1)t_1}dt_1.\int_0^\infty e^{-(s_2-a_2)t_2}dt_2\int_0^\infty e^{-s_3t_3}\ dt_3$$

$$L_3\{e^{a_1t_{1+}a_1t_1}\} = \frac{1}{(s_1-a_1)(s_2-a_2)s_3}$$

Similarly

$L_n\{e^{a_1t_{1+}a_2t_2}\} = \frac{1}{(s_1-a_1)(s_2-a_2)s_3\ldots s_n}$,

$$L_n\{e^{a_1t_{1+}a_2t_2+a_3t_3}\} = \frac{1}{(s_1-a_1)(s_2-a_2)(s_3-a_3)s_4\ldots s_n}$$

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Similarly

$L_n\{e^{a_1t_{1+}a_2t_2+\cdots+a_nt_n}\} = \frac{1}{(s_1-a_1)(s_2-a_2)\ldots(s_n-a_n)}$, for $s_i > a_i$ for i= 1, 2,3…n

5. $L_2\{\sin(a_1t_1+a_2t_2)\} = \int_0^\infty\int_0^\infty e^{-s_1t_1-s_2t_2}.\sin(a_1t_{1+}a_2t_2)\,dt_1dt_2$

$$= \int_0^\infty\int_0^\infty e^{-s_1t_1-s_2t_2}.\sin(a_1t_1).\cos(a_2t_2)\,dt_1dt_2 + \int_0^\infty\int_0^\infty e^{-s_1t_1-s_2t_2}.\cos(a_1t_1).\,sin(a_2t_2)\,dt_1dt_2$$

$$=\int_0^\infty e^{-s_1t_1}\sin(a_1t_1)dt_1\int_0^\infty e^{-s_2t_2}\cos(a_2t_2)\,dt_2 + \int_0^\infty e^{-s_1t_1}\cos(a_1t_1)dt_1\int_0^\infty e^{-s_2t_2}.\,sin(a_2t_2)\,dt_2$$

$$= \frac{a_1}{(s_1^2+a_1^2)}.\frac{s_2}{(s_2^2+a_2^2)} + \frac{s_1}{(s_1^2+a_1^2)}.\frac{a_2}{(s_2^2+a_2^2)}$$

$$= \frac{a_1s_2+a_2s_1}{(s_1^2+a_1^2)(s_2^2+a_2^2)}$$

Now $L_3\{\sin(a_1t_1+a_2t_2)\} = \int_0^\infty\int_0^\infty\int_0^\infty e^{-s_1t_1-s_2t_2-s_3t_3}.\sin(a_1t_1+a_2t_2)\,dt_1dt_2dt_3$

$$= \int_0^\infty\int_0^\infty\int_0^\infty e^{-s_1t_1-s_2t_2-s_3t_3}.\sin(a_1t_1)cos(a_2t_2)\,dt_1dt_2dt_3 + \int_0^\infty\int_0^\infty\int_0^\infty e^{-s_1t_1-s_2t_2-s_3t_3}.\cos(a_1t_1).\sin(a_2t_2)\,dt_1dt_2dt_3$$

$$= \int_0^\infty e^{-s_1t_1}\sin(a_1t_1)dt_1\int_0^\infty e^{-s_2t_2}cos(a_2t_2)dt_2\int_0^\infty e^{-s_3t_3}\,dt_3 + \int_0^\infty e^{-s_1t_1}\cos(a_1t_1)dt_1\int_0^\infty e^{-s_2t_2}sin(a_2t_2)dt_2\int_0^\infty e^{-s_3t_3}\,dt_3$$

$$=\frac{a_1}{(s_1^2+a_1^2)}.\frac{s_2}{(s_2^2+a_2^2)}.\frac{1}{s_3} + \frac{s_1}{(s_1^2+a_1^2)}.\frac{a_2}{(s_2^2+a_2^2)}.\frac{1}{s_3}$$

$$= \frac{a_1s_2+a_2s_1}{(s_1^2+a_1^2)(s_2^2+a_2^2).s_3}$$

Similarly

$$L_n\{\sin(a_1t_1+a_2t_2)\} = \frac{a_1s_2+a_2s_1}{(s_1^2+a_1^2)(s_2^2+a_2^2).s_3\ldots s_n}$$

6. $L_2\{\cos(a_1t_1+a_2t_2)\} = \int_0^\infty \int_0^\infty e^{-s_1t_1-s_2t_2}.\cos(a_1t_{1+}a_2t_2)\,dt_1dt_2$

$$= \int_0^\infty \int_0^\infty e^{-s_1t_1-s_2t_2}.\cos(a_1t_1).\cos(a_2t_2)\,dt_1dt_2 - \int_0^\infty \int_0^\infty e^{-s_1t_1-s_2t_2}.\sin(a_1t_1).sin(a_2t_2)\,dt_1dt_2$$

$$=\int_0^\infty e^{-s_1t_1}\cos(a_1t_1)dt_1 \int_0^\infty e^{-s_2t_2}\cos(a_2t_2)\,dt_2 - \int_0^\infty e^{-s_1t_1}\sin(a_1t_1)dt_1 \int_0^\infty e^{-s_2t_2}.sin(a_2t_2)\,dt_2$$

$$= \frac{s_1}{(s_1^2+a_1^2)}.\frac{s_2}{(s_2^2+a_2^2)} - \frac{a_1}{(s_1^2+a_1^2)}.\frac{a_2}{(s_2^2+a_2^2)}$$

$$= \frac{s_1s_2 - a_1a_2}{(s_1^2+a_1^2)(s_2^2+a_2^2)}$$

Now

$$L_3\{\cos(a_1t_1+a_2t_2)\} = \int_0^\infty \int_0^\infty \int_0^\infty e^{-s_1t_1-s_2t_2-s_3t_3}.\cos(a_1t_1+a_2t_2)\,dt_1dt_2dt_3$$

$$= \int_0^\infty \int_0^\infty \int_0^\infty e^{-s_1t_1-s_2t_2-s_3t_3}.\cos(a_1t_1)cos(a_2t_2)\,dt_1dt_2dt_3 - \int_0^\infty \int_0^\infty \int_0^\infty e^{-s_1t_1-s_2t_2-s_3t_3}.\sin(a_1t_1).\sin(a_2t_2)\,dt_1dt_2dt_3$$

$$= \int_0^\infty e^{-s_1t_1}\cos(a_1t_1)dt_1 \int_0^\infty e^{-s_2t_2}cos(a_2t_2)dt_2 \int_0^\infty e^{-s_3t_3}\,dt_3 - \int_0^\infty e^{-s_1t_1}\sin(a_1t_1)dt_1 \int_0^\infty e^{-s_2t_2}sin(a_2t_2)dt_2 \int_0^\infty e^{-s_3t_3}\,dt_3$$

$$=\frac{s_1}{(s_1^2+a_1^2)}.\frac{s_2}{(s_2^2+a_2^2)}.\frac{1}{s_3} - \frac{a_1}{(s_1^2+a_1^2)}.\frac{a_2}{(s_2^2+a_2^2)}.\frac{1}{s_3}$$

$$= \frac{s_1s_2-a_1a_2}{(s_1^2+a_1^2)(s_2^2+a_2^2).s_3}$$

Similarly

$$L_n\{\cos(a_1t_1+a_2t_2)\} = \frac{s_1s_2 - a_1a_2}{(s_1^2+a_1^2)(s_2^2+a_2^2).s_3 \ldots s_n}$$

7. $L_2\{\cosh(a_1t_1+a_2t_2)\} = \int_0^\infty \int_0^\infty e^{-s_1t_1-s_2t_2}.\cosh(a_1t_{1+}a_2t_2)\,dt_1dt_2$

Where h represents hyperbolic

$L_2\{\cosh(a_1t_1+a_2t_2)\} = \int_0^\infty \int_0^\infty e^{-s_1t_1-s_2t_2}.\cosh(a_1t_{1+}a_2t_2)\,dt_1dt_2$

$$=\int_0^\infty \int_0^\infty e^{-s_1t_1-s_2t_2}.\frac{e^{(a_1t_1+a_2t_2)}+e^{-(a_1t_1+a_2t_2)}}{2}\,dt_1dt_2$$

$$= \frac{1}{2}\int_0^\infty \int_0^\infty e^{-s_1t_1-s_2t_2}e^{(a_1t_{1+}a_2t_2)}\,dt_1dt_2 + \frac{1}{2}\int_0^\infty \int_0^\infty e^{-s_1t_1-s_2t_2}e^{-(a_1t_1+a_2t_2)}\,dt_1dt_2$$

$$= \frac{1}{2}\int_0^\infty \int_0^\infty e^{-(s_1-a_1)t_1-(s_2-a_2)t_2}\, dt_1 dt_2 +$$

$$\frac{1}{2}\int_0^\infty \int_0^\infty e^{-(s_1+a_1)t_1-(s_2+a_2)t_2}\, dt_1 dt_2$$

$$= \frac{1}{2}\left[\frac{1}{(s_1-a_1).(s_2-a_2)} + \frac{1}{(s_1+a_1).(s_2+a_2)}\right]$$

$$= \frac{1}{2}\left[\frac{(s_1+a_1).(s_2+a_2)+(s_1-a_1).(s_2-a_2)}{(s_1^2+a_1^2)(s_2^2+a_2^2)}\right]$$

$$= \frac{1}{2}\left[\frac{s_1\, s_2+s_1\, a_2+a_1\, s_2+a_1\, a_2+s_1\, s_2-s_1\, a_2-a_1\, s_2+a_1\, a_2}{(s_1^2+a_1^2)(s_2^2+a_2^2)}\right]$$

$$= \frac{s_1\, s_2+a_1\, a_2}{(s_1^2+a_1^2)(s_2^2+a_2^2)}$$

Now

$$L_3\{\cosh(a_1t_1+a_2t_2)\} = \int_0^\infty \int_0^\infty \int_0^\infty e^{-s_1t_1-s_2t_2-s_3t_3}.\cosh(a_1t_1+a_2t_2)\, dt_1 dt_2$$

$$\int_0^\infty \int_0^\infty \int_0^\infty e^{-s_1t_1-s_2t_2-s_3t_3}.\frac{e^{(a_1t_1+a_2t_2)}+e^{-(a_1t_1+a_2t_2)}}{2}\, dt_1 dt_2$$

$$= \frac{1}{2}\int_0^\infty \int_0^\infty \int_0^\infty e^{-s_1t_1-s_2t_2-s_3t_3}.e^{(a_1t_1+a_2t_2)}\, dt_1 dt_2 +$$

$$\frac{1}{2}\int_0^\infty \int_0^\infty \int_0^\infty e^{-s_1t_1-s_2t_2-s_3t_3}.e^{-(a_1t_1+a_2t_2)}\, dt_1 dt_2$$

$$= \frac{1}{2}\int_0^\infty \int_0^\infty \int_0^\infty e^{-(s_1-a_1)t_1-(s_2-a_2)t_2-s_3t_3}\, dt_1 dt_2 +$$

$$\frac{1}{2}\int_0^\infty \int_0^\infty \int_0^\infty e^{-(s_1+a_1)t_1\mp a_2)t_2-s_3t_3}\, dt_1 dt_2$$

$$= \frac{1}{2}\left[\frac{1}{(s_1-a_1).(s_2-a_2)s_3} + \frac{1}{(s_1+a_1).(s_2+a_2)s_3}\right]$$

$$= \frac{1}{2}\left[\frac{(s_1+a_1).(s_2+a_2)+(s_1-a_1).(s_2-a_2)}{(s_1^2+a_1^2)(s_2^2+a_2^2)s_3}\right]$$

$$= \frac{1}{2}\left[\frac{s_1\, s_2+s_1\, a_2+a_1\, s_2+a_1\, a_2+s_1\, s_2-s_1\, a_2-a_1\, s_2+a_1\, a_2}{(s_1^2+a_1^2)(s_2^2+a_2^2)s_3}\right]$$

$$= \frac{s_1\, s_2+a_1\, a_2}{(s_1^2+a_1^2)(s_2^2+a_2^2)s_3}$$

Similarly

$$L_n\{\cosh(a_1t_1+a_2t_2)\} = \frac{s_1\, s_2+a_1\, a_2}{(s_1^2+a_1^2)(s_2^2+a_2^2)s_3 \ldots s_n}$$

8. $L_2\{\sinh(a_1t_1+a_2t_2)\} = \int_0^\infty \int_0^\infty e^{-s_1t_1-s_2t_2}.\sinh(a_1t_1+a_2t_2)\, dt_1 dt_2$
   Where h represents hyperbolic
   $L_2\{\sinh(a_1t_1+a_2t_2)\} = \int_0^\infty \int_0^\infty e^{-s_1t_1-s_2t_2}.\sinh(a_1t_1+a_2t_2)\, dt_1 dt_2$

$$=\int_0^\infty \int_0^\infty e^{-s_1t_1-s_2t_2}.\frac{e^{(a_1t_1+a_2t_2)}-e^{-(a_1t_1+a_2t_2)}}{2}\,dt_1dt_2$$

$$=\frac{1}{2}\int_0^\infty \int_0^\infty e^{-s_1t_1-s_2t_2}e^{(a_1t_1+a_2t_2)}\,dt_1dt_2 - \frac{1}{2}\int_0^\infty \int_0^\infty e^{-s_1t_1-s_2t_2}e^{-(a_1t_1+a_2t_2)}\,dt_1dt_2$$

$$=\frac{1}{2}\int_0^\infty \int_0^\infty e^{-(s_1-a_1)t_1-(s_2-a_2)t_2}\,dt_1dt_2 - \frac{1}{2}\int_0^\infty \int_0^\infty e^{-(s_1+a_1)t_1-(s_2+a_2)t_2}\,dt_1dt_2$$

$$=\frac{1}{2}\left[\frac{1}{(s_1-a_1).(s_2-a_2)}-\frac{1}{(s_1+a_1).(s_2+a_2)}\right]$$

$$=\frac{1}{2}\left[\frac{(s_1+a_1).(s_2+a_2)-(s_1-a_1).(s_2-a_2)}{(s_1^2+a_1^2)(s_2^2+a_2^2)}\right]$$

$$=\frac{1}{2}\left[\frac{s_1\,s_2+s_1\,a_2+a_1\,s_2+a_1\,a_2-s_1\,s_2+s_1\,a_2+a_1\,s_2-a_1\,a_2}{(s_1^2+a_1^2)(s_2^2+a_2^2)}\right]$$

$$=\frac{s_1\,a_2+a_1\,s_2}{(s_1^2+a_1^2)(s_2^2+a_2^2)}$$

Now

$$L_3\{\sinh(a_1t_1+a_2t_2)\} = \int_0^\infty \int_0^\infty \int_0^\infty e^{-s_1t_1-s_2t_2-s_3t_3}.\sinh(a_1t_1+a_2t_2)\,dt_1dt_2$$

$$=\int_0^\infty \int_0^\infty \int_0^\infty e^{-s_1t_1-s_2t_2-s_3t_3}.\frac{e^{(a_1t_1+a_2t_2)}-e^{-(a_1t_1+a_2t_2)}}{2}\,dt_1dt_2$$

$$=\frac{1}{2}\int_0^\infty \int_0^\infty \int_0^\infty e^{-s_1t_1-s_2t_2-s_3t_3}.e^{(a_1t_1+a_2t_2)}\,dt_1dt_2 - \frac{1}{2}\int_0^\infty \int_0^\infty \int_0^\infty e^{-s_1t_1-s_2t_2-s_3t_3}.e^{-(a_1t_1+a_2t_2)}\,dt_1dt_2$$

$$=\frac{1}{2}\int_0^\infty \int_0^\infty \int_0^\infty e^{-(s_1-a_1)t_1-(s_2-a_2)t_2-s_3t_3}\,dt_1dt_2 - \frac{1}{2}\int_0^\infty \int_0^\infty \int_0^\infty e^{-(s_1+a_1)t_1\mp a_2)t_2-s_3t_3}\,dt_1dt_2$$

$$=\frac{1}{2}\left[\frac{1}{(s_1-a_1).(s_2-a_2)s_3}-\frac{1}{(s_1+a_1).(s_2+a_2)s_3}\right]$$

$$=\frac{1}{2}\left[\frac{(s_1+a_1).(s_2+a_2)-(s_1-a_1).(s_2-a_2)}{(s_1^2+a_1^2)(s_2^2+a_2^2)s_3}\right]$$

$$=\frac{1}{2}\left[\frac{s_1\,s_2+s_1\,a_2+a_1\,s_2+a_1\,a_2-s_1\,s_2+s_1\,a_2+a_1\,s_2-a_1\,a_2}{(s_1^2+a_1^2)(s_2^2+a_2^2)s_3}\right]$$

$$=\frac{s_1\,a_2+a_1\,s_2}{(s_1^2+a_1^2)(s_2^2+a_2^2)s_3}$$

Similarly $L_n\{\sinh(a_1t_1+a_2t_2)\} = \frac{s_1\,a_2+a_1\,s_2}{(s_1^2+a_1^2)(s_2^2+a_2^2)s_3\ldots s_n}$

6.7: Now we have N-dimensional Laplace Transform Table-1 for two variables:

S.No.	Function	Laplace Transform
1	$L_n\{1\}$	$\frac{1}{s_1}\frac{1}{s_2}\frac{1}{s_3}\dots\frac{1}{s_n}$
2	$L_n\{t_1\}$	$\frac{1}{s_1^2}.\frac{1}{s_2}\frac{1}{s_3\dots s_n}$
3	$L_n\{t_1+t_2\}$	$\frac{1}{s_1^2}.\frac{1}{s_2}\frac{1}{s_3\dots s_n}+\frac{1}{s_2^2}.\frac{1}{s_1}\frac{1}{s_3\dots s_n}$
4	$L_n\{(t_1+t_2)^2\}$	$\frac{2!}{s_1 s_2 s_3\dots s_n}\left[\frac{1}{s_1^2}+\frac{1}{s_2^2}+\frac{1}{s_1 s_2}\right] or \frac{2!\,(s_1^2+s_2^2+s_1 s_2)}{s_1^3 s_2^3 s_3\dots s_n}$
5	$L_n\{e^{a_1 t_1+a_2 t_2}\}$	$\frac{1}{(s_1-a_1)(s_2-a_2)s_3\dots s_n}$, where $s_i > a_i$ for i= 1,2
6	$L_n\{\sin(a_1 t_1+a_2 t_2)\}$	$\frac{a_1 s_2+a_2 s_1}{(s_1^2+a_1^2)(s_2^2+a_2^2).s_3\dots s_n}$
7	$L_n\{\cos(a_1 t_1+a_2 t_2)\}$	$\frac{s_1 s_2-a_1 a_2}{(s_1^2+a_1^2)(s_2^2+a_2^2).s_3\dots s_n}$
8	$L_n\{\sinh(a_1 t_1+a_2 t_2)\}$	$\frac{s_1\,a_2+a_1\,s_2}{(s_1^2+a_1^2)(s_2^2+a_2^2)s_3\dots s_n}$
9	$L_n\{\cosh(a_1 t_1+a_2 t_2)\}$	$\frac{s_1\,s_2+a_1\,a_2}{(s_1^2+a_1^2)(s_2^2+a_2^2)s_3\dots s_n}$

Remarks: We can also develop N-dimensional Laplace table for n-Variables , few n-variable functions are here

$$L_n\{e^{a_1 t_1+a_2 t_2+\dots+a_n t_n}\}=\frac{1}{(s_1-a_1)(s_2-a_2)\dots(s_n-a_n)}$$, for $s_i > a_i$ for i= 1, 2,3…n

$$L_n\{t_1+t_2+t_3\dots+t_n\}=\frac{1}{s_1^2}.\frac{1}{s_2}\frac{1}{s_3\dots s_n}+\frac{1}{s_2^2}.\frac{1}{s_1}\frac{1}{s_3\dots s_n}+\frac{1}{s_3^2}.\frac{1}{s_2}\frac{1}{s_1\dots s_n}+\dots+.\frac{1}{s_1 s_2 s_3\dots s_n^2}$$

6.8 First Translating Shifting Property: (For Two variables):

If $L_n\{t_1+t_2\}=\frac{1}{s_1^2}.\frac{1}{s_2}\frac{1}{s_3\dots s_n}+\frac{1}{s_2^2}.\frac{1}{s_1}\frac{1}{s_3\dots s_n}$ then

$$L_n\{e^{a_1 t_1+a_2 t_2}(t_1+t_2)\}=\frac{(s_2-a_2)-(s_1-a_1)}{(s_1-a_1)^2.(s_2-a_2)^2}$$

Proof:

First we have found

$$L_2\{e^{a_1t_1+a_2t_2}(t_1+t_2)\} = \int_0^\infty\int_0^\infty e^{-s_1t_1-s_2t_2}.e^{a_1t_1+a_2t_2}(t_1+t_2)dt_1dt_2$$

$$= \int_0^\infty\int_0^\infty e^{-s_1t_1-s_2t_2}.e^{a_1t_1+a_2t_2}.t_1dt_1dt_2 + \int_0^\infty\int_0^\infty e^{-s_1t_1-s_2t_2}.e^{a_1t_1+a_2t_2}t_2.dt_1dt_2$$

$$= \int_0^\infty e^{-(s_1-a_1)t_1}.t_1dt_1.\int_0^\infty e^{-(s_2-a_2)t_2}dt_2+\int_0^\infty e^{-(s_1-a_1)t_1}dt_1.\int_0^\infty e^{-(s_2-a_2)t_2}t_2dt_2$$

$$=\left[t_1.\left(-\frac{e^{-(s_1-a_1)t_1}}{(s_1-a_1)}\right)-(1).\left(+\frac{e^{-(s_1-a_1)t_1}}{(s_1-a_1)^2}\right)\right]_{t=0}^{t=\infty}.\left[\left(-\frac{e^{-(s_2-a_2)t_2}}{(s_2-a_2))}\right)\right]_{t=0}^{t=\infty}+$$
$$\left[\left(-\frac{e^{-(s_1-a_1)t_1}}{(s_1-a_1)}\right)\right]_{t=0}^{t=\infty}\left[t_2.\left(-\frac{e^{-(s_2-a_2)t_2}}{(s_2-a_2)}\right)-(1).\left(+\frac{e^{-(s_2-a_2)t_2}}{(s_2-a_2)^2}\right)\right]_{t=0}^{t=\infty}$$

$$=\left[(0-0)-(0-\frac{1}{(s_1-a_1)^2})\right].\left(\frac{1}{(s_2-a_2)}\right)+\left(\frac{1}{(s_1-a_1)}\right)\left[(0-0)-(0-\frac{1}{(s_2-a_2)^2})\right]$$

$$=\frac{1}{(s_1-a_1)^2}.\frac{1}{(s_2-a_2)}+\left(\frac{1}{(s_1-a_1)}\right)\frac{1}{(s_2-a_2)^2}$$

$$=\frac{1}{(s_2-a_2)}.\frac{1}{(s_2-a_2)}\left[\frac{1}{(s_1-a_1)}-\frac{1}{(s_2-a_2)}\right]\text{or }\frac{(s_2-a_2)-(s_1-a_1)}{(s_1-a_1)^2.(s_2-a_2)^2}$$

Therefore

$$L_n\{e^{a_1t_1+a_2t_2}(t_1+t_2)\} = \frac{(s_2-a_2)-(s_1-a_1)}{(s_1-a_1)^2.(s_2-a_2)^2}$$

Similarly we can evaluate second shifting and scale property.

7. Result and Discussion: The Laplace transform is a mathematical technique used to analyze and solve linear time-invariant systems. It converts a function of time, f(t), into a function of a complex variable, F(s), where 's' is a complex number. The one-dimensional Laplace transform is defined as follows: F(s) = L[f(t)] = ∫[0, ∞] e^(-st) f(t) dt n-Dimensional Laplace Transform: Extending the concept to n-dimensional systems, where 'n' can represent multiple variables (e.g., time and space), leads to the n-dimensional Laplace transform.

We believe that these results will further enhance the use of 2 and n-dimensional Laplace transformations and help further development of more theoretical results. Even though multi-dimensional Laplace transformation have been studied extensively since the early 1920s, or so, still a table of three on N-dimensional Laplace transforms is not available. To fill this gap much work is left to be done. To this end, we have developed several new results on 2-dimensional Laplace transformations as well as inverse Laplace transformation and many more are still under our investigation. A successful completion of this task will be a significant endeavor, which will be extremely beneficial to the further research in Applied Mathematics, Engineering and Physical Sciences. Specially, by the use of multi-dimensional Laplace transformations a PDE and its associated boundary conditions can be transformed into an algebraic equation in n independent variables, this algebraic equation can be solved to obtain the desired solution. We believe that these results will further enhance the use of 2 dimensional Laplace Transformation and help further development of more theoretical results. Several initial boundary value problems (IBVPs) characterized by Non-Homogenous linear partial differential equations (PDEs) are explicitly. So in this regards we explain 2 dimensional tables (3.4) and also develop a table (5.3.1) for n-dimensional Laplace transform for n variables. By using 2 dimensional Laplace Transform we analysis a transport equation and heat equation make solution $\boldsymbol{u(x,y)} = \left[\boldsymbol{f}\left(\boldsymbol{x} - \frac{\boldsymbol{a}}{\boldsymbol{b}}\boldsymbol{y}\right)\right]$ (4.1) and $u(x,t) = e^{-\frac{1}{\lambda}t}$ (5.4) . In (4.2) and (4.3) equation we also develop transport equation in particular and spatial cases, where we got some unique and standard result for transport equation those are helpful to develop signal system and by heat conduction equation we see the temperature is length independent which is more useful to develop new technology to maintain fog in cars during winter and also in cooling aspects.

8. Conclusion: In this paper we study about 2 and n dimensional Laplace Transform because one and two dimensional Laplace transform material is easily available but n dimensional Laplace table is not available so far, therefore we develop a n- dimensional Laplace table (5.3.1) for two variable, there is more work to left for research to develop a n-dimensional table for multivariable functions.

In equation 4.1 ,4.2 and 4.3 we have some results for transport equation which are useful to explain The transport equation, as u(x, y) = x - y,(for a =1 and b= 1,in equation 4.1) is a mathematical expression that describes the flow or transport of a quantity (u) in two dimensions, typically in the context of fluid dynamics or transport phenomena. It may not be a direct representation of signal and system theory, but it can be related to real-life scenarios and signal processing in the following ways:

Image Processing: In image processing, the transport equation can be used to describe the movement or transformation of intensity values within an image. For example, it could represent the shifting of pixels in a 2D image, which is a common operation in various image processing algorithms.

Optical Flow: Optical flow is a technique used in computer vision to estimate the motion of objects in a video sequence. The transport equation can be adapted to describe the flow of image features from one frame to another, aiding in object tracking and motion analysis.

Heat Transfer: In the context of heat transfer, the transport equation can be applied to understand how heat propagates in a 2D domain. It can be useful for modeling temperature distribution in materials or fluids.

Transport of Particles: The transport equation can also represent the movement of particles or substances in a 2D space. For instance, it can describe the dispersion of pollutants in the atmosphere or the diffusion of chemicals in a biological system.

In signal and system theory, the transport equation itself is not a standard concept, but the principles of transport and flow can be applied to understand how signals propagate or evolve in a system. Real-life applications may include:

Audio Processing: Understanding how sound waves travel through different media or how signals move through audio processing filters can be related to the concept of signal transport.

Data Transmission: In the context of data communication, signal propagation along transmission lines or through wireless channels can be analyzed using principles related to signal transport.

Control Systems: The way control signals or feedback information travel through a control system to regulate a process can be seen as an application of signal transport. While the transport equation itself is not directly applied in signal and system theory, the underlying concept of transport and propagation is relevant in understanding various real-world phenomena and can be analogously related to signal behavior in certain applications.

In this paper we also found a relation in equation 5.4 that is $u(x,t) = e^{-\frac{1}{\lambda}t}$

Which explain flow of temperature in rod and got a length independent temperature relation, a temperature rod with length-independent temperature decay is an interesting concept that can have practical applications in various fields. Here are a few examples of where such a rod might be applied:

In the construction and building industry, length-independent temperature decay can be beneficial for designing effective thermal insulation materials. This property would ensure that heat doesn't dissipate more rapidly over longer sections of the material, leading to improved energy efficiency in buildings and also in the field of cryogenics and cryopreservation, maintaining a consistent and low temperature across a sample is crucial. Length-independent temperature decay could help ensure that samples, such as biological tissues or cells, are uniformly and efficiently preserved without variations in temperature along their length, the length-independent temperature decay can be advantageous in the design of heat exchangers, allowing for a more uniform transfer of thermal energy in various industrial processes.

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