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TYPE 1I HALF-LOGISTIC EXPONENTIATED INVERSE EXPONENTIAL DISTRIBUTION-G FAMILYY OF DISTRIBUTIONS WITH APPLICATIONS TO BIOMEDICAL DATA

BY

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Abstract : This paper introduced a new distribution called type 11 Half-Logistic Exponentiated inverse Exponential (**TIIHLEtIEx**) distribution established from type 11 Half-Logistic Exponentiated G-family of distributions. Some mathematical properties like Probability Weighted Moments**,** moments, Moment Generating Function, Reliability function, Quantile Function investigated. The expressions of order statistics are derived. Parameters of the derived distribution are obtained using maximum likelihood method. We compared the fits of type 11 Half-Logistic Exponentiated inverse Exponential (**TIIHLEtIEx)** model with some comparator models, it found that **TIIHLEtIEx** perform better in terms robustness, flexibility and applicability than all the comparators distributions model.

Keywords: Probability Weighted Moments**,** moments, Moment Generating Function, Reliability function, Quantile Function, order statistics, Maximum Likelihood.

INTRODUCTION

The creation of novel, generalized statistical models is a significant field of research in distribution theory. Such distributions, which are particularly useful in forecasting and simulating real-world phenomena, are widely available in the literature. Over the past few decades, a variety of classical distributions have been extensively utilized to model data in a variety of practical fields, such as biomedical analysis, reliability engineering, economics, forecasting, astronomy, demography, and insurance.

Significant work has gone into creating common probability distributions and the associated statistical techniques. Classical distributions frequently do not offer an appropriate match to real-world data sets. As a result, various families of continuous distributions have been constructed in the literature, using one or more parameters to produce new distributions. These generalized classes of distributions can accommodate both monotonic and non-monotonic hazard failure rate shapes and are particularly adaptable. Many of the developed families were examined in various contexts and found to produce more application versatility. For example, Alzaatreh et al., (2013) defined the T-X family of distribution, Yousof et al., (2018), obtained a generalized version of the Marshall-Olkin-G family of distributions, Nasir et al., (2017) proposed a new family of distributions called the generalized Burr-G family of distributions, Alizadeh et al., (2015) introduced the Kumaraswamy Marshall-Olkin family of distributions, Afify et al., (2016) defined the Kumaraswamy Transmuted-G family of distributions, Alizadeh et al., (2017) defined the generalized odd generalized exponential-G family of distributions, Silva et al., (2019) defined a new class of distributions that extends the Kumaraswamy-G family of distributions, Al-Shomrani et al., (2016) proposed the Topp-Leone-G family of distributions, Yousof et al., (2017) defined a new family of distributions called transmuted Topp-Leone-G family of distributions, Sanusi, et al., (2020) proposed a new family of distributions called Toppleone Exponential-G family of distributions, Falgore and Doguwa (2020) proposed a new generator of continuous distributions with four positive parameters called the Kumaraswamy-Odd Rayleigh-G family of distributions, Ibrahim et al., (2020a) developed Topp-Leone exponentiated-G family of distributions, Bello et al., (2020a) proposed Type I half logistic exponentiated-G family of distributions, Bello et al., (2020b) proposed Type II half logistic exponentiated-G family of distributions.

The positive half of the logistic distribution is represented by a probability distribution called the half logistic distribution. It has a symmetric, bell-shaped curve that is bounded at zero and is a continuous probability distribution. The distribution is frequently used in probability theory and statistics to describe positive continuous variables. When working with positive data that displays a symmetric pattern and has a natural bottom bound at zero, the half logistic distribution is frequently employed as a modeling tool. It can be used in disciplines like dependability analysis, finance, and economics.

The half logistic-G distribution can be used to model a variety of data, including survival times, waiting times, and economic data. It has been shown to be more flexible than the half logistic distribution, and it can be used to model data with a wider range of shapes. Using the half-logistic distribution, Cordeiro et al., (2017) introduced the type I half-logistic family of distributions, Adepoju et al., (2021) introduced the type I half logistic-T-G family of distributions, Moakofi et al., (2021) derived type II exponentiated half-logistic Topp-Leone Marshall-Olkin-G family of distributions.

The main objective this paper is to developed a novel model that more robust and flexible in modeling real life data set and also to evaluate the impact of introducing additional parameters to the distribution and how this affects its flexibility, applicability, and overall effectiveness. By comparing the baseline distribution with the new distributions that have additional parameters, we can gain insights into how these modifications enhance or alter the distribution's ability to fit real-life data and address various modeling challenges.

NEED OF THE STUDY.

This study presents two distributions developed from families of distributions presented by Bello et al., (2020 and 2021). In addition to providing an option to other models for modeling lifespan data sets originating from the biomedical area, these distributions will boost the flexibility and resilience of the fundamental inverse exponential distribution.

3.1Population and Sample

The Population of this research work was the simulation of 1000 replicates from the new models usingquantile functions and the application of the new models to four (4) different real-life data sets.

The Sample of this research work will contain selection of sample sizes for n = 20, 50, 100, 150,200, and 250 through simulation study and by uses of daily real-life data sets.

3.2 Data and Sources of Data

Data that will be used for this research work will be collected in two stages. The first Stage will be through a simulation study and the second stage of data collection is secondary data which will be extract from already published articles..

3.3 Theoretical framework

Modeling random variables that can have any value within a range is done using continuous probability distributions. A continuous random variable is, for instance, the height of a randomly chosen individual. Between 0 and infinity, it can take on any value. An infinitesimal interval of values within a continuous random variable is given a probability via a probability density function, which represents the probability distribution of that interval. Examples of continuous probability distribution are Exponential, Inverse exponential, Uniform, Normal, Weibull, Inverse Weibull, Rayleigh, Inverse Rayleigh, Log-normal, Gamma, Logistic, Half-logistic, Frechet, Gumbel, Beta, Kumaraswamy, Pareto, Triangular, Nakagami, Chen, Burr, Topp-Leone, Cauchy, Lomax, Inverse Lomax, Dagum, Gompertz, Chi-square, Student-t, Lindley and Birnbaum-Saunders distributions among others. These probability distributions have distinct features and properties which made them find their ways in real-life applications. Some of the properties of probability distributions are quantile function, survival function, hazard function, moments, mean, moment generating function, median, skewness, kurtosis, reverse hazard function, odds function, cumulative hazard function, probability weighted moment, entropies, and so on.

Many different disciplines, such as statistics, machine learning, and finance, make use of probability distributions. They are applied in the planning of experiments, the formulation of hypotheses, and the formulation of predictions about future events.

RESEARCH METHODOLOGY

2.1 Type II Half-Logistic Exponentiated-G Family of Distributions

Bello *et al.* (2021) presented the Type II Half-Logistic Exponentiated-G Family of distributions, which provides improved flexibility and effectively captures the characteristics of various datasets. Bello et al. (2021) also provided the cumulative distribution function (cdf) for the Type II Half-Logistic Exponentiated-G Family of distributions in their study.

$$F_{TIIHLEt\_G}(x;\lambda,\alpha,\boldsymbol{\beta}) = \frac{2H^{\alpha\lambda}(x;\boldsymbol{\beta})}{\left[1+H^{\alpha\lambda}(x;\boldsymbol{\beta})\right]}, \ x>0, \lambda,\alpha>0 \ \text{ and } \boldsymbol{\beta} \text{ is parameter vector} \quad (1)$$

Where λ and α are shape parameters H(x;**β**) is the cdf of the baseline model. The corresponding pdf is given by

$$f_{TIIHLEt-G}(x;\lambda,\alpha,\boldsymbol{\beta}) = \frac{2\lambda\alpha h(x;\boldsymbol{\beta})H^{\alpha-1}(x;\boldsymbol{\beta})\left[H^{\alpha(\lambda-1)}(x;\boldsymbol{\beta})\right]}{\left[1+H^{\alpha\lambda}(x;\boldsymbol{\beta})\right]^2}, \ x>0, \ \lambda,\alpha>0 \quad (2)$$

2.2 Inverse Exponential Distribution

Suppose that X is a continuous random variable, then X is said to have followed inverse exponential distribution if it cdf and pdf are expressed as;

$$H(x) = e^{-\left(\frac{\theta}{x}\right)} \quad (3)$$

$$h(x) = \left(\frac{\theta}{x^2}\right)e^{-\left(\frac{\theta}{x}\right)}, \quad x>0, \theta>0 \quad (4)$$

where θ is a scale parameter

The random variable X is said to have a TIIHLEtIEx distribution, if its cdf is obtained by inserting equation (3) in equation (1) as follows

$$F(x,\alpha,\lambda,\theta)=\frac{2\left[e^{-\left(\frac{\theta}{x}\right)}\right]^{\alpha\lambda}}{1+\left[e^{-\left(\frac{\theta}{x}\right)}\right]^{\alpha\lambda}} \qquad (5)$$

on differentiating (5), the pdf of TIIHLEtIEx is as follows

$$f(x,\alpha,\lambda,\theta)=\frac{2\alpha\lambda\left(\frac{\theta}{x^2}\right)e^{-\left(\frac{\theta}{x}\right)}\left[e^{-\left(\frac{\theta}{x}\right)}\right]^{\alpha-1}\left[e^{-\left(\frac{\theta}{x}\right)}\right]^{\alpha(\lambda-1)}}{\left[1+\left[e^{-\left(\frac{\theta}{x}\right)}\right]^{\alpha\lambda}\right]^2} \qquad (6)$$

3.12 Expansion of Density for TIIHLEtIEx Distribution

In this section, a useful expansion of the PDF and CDF for TIIHLEtIEx Distribution are provided.Since the generalized binomial series is:

$$\left[1+z\right]^{-b}=\sum_{i=0}^{\infty}(-1)^i\binom{b+i-1}{i}z^i \qquad (7)$$

Using the last term in the equation (6) in relation to equation (7), we have

$$\left[1+\left[e^{-\frac{\theta}{x}}\right]^{\alpha\lambda}\right]^{-2}=\sum_{i=0}^{\infty}(-1)^i\binom{1+i}{i}\left[e^{-\frac{\theta}{x}}\right]^{\alpha\lambda i} \qquad (8)$$

Substituting equation (8) into equation (6), we have

$$f(x,\alpha,\lambda,\theta)=2\alpha\lambda\left(\frac{\theta}{x^2}\right)\sum_{i=0}^{\infty}(-1)^i\binom{1+i}{i}\left[e^{-\frac{\theta}{x}}\right]^{\alpha\lambda(i+1)} \qquad (9)$$

The equation (9) above can be rewrite, so we have

$$f(x,\alpha,\lambda,\theta)=\sum_{i=0}^{\infty}\psi\frac{\theta}{x^2}\left[e^{-\left(\frac{\theta}{x}\right)}\right]^{\alpha\lambda(i+1)} \qquad (10)$$

where $\psi=2\alpha\lambda(-1)^i\binom{i+1}{i}$

Equation (10) is the important representation of the pdf of TIIHLEtIEx distribution.

Also, an expansion for the CDF, using the binomial expansion $\left[F(x,\alpha,\lambda,\theta)\right]^h$ where h is an integer, lead to:

$$\left[F(x,\alpha,\lambda,\theta)\right]^h=\underbrace{\left[2\left[e^{-\frac{\theta}{x}}\right]^{\alpha\lambda}\right]^h}_{A}\underbrace{\left[1+\left[e^{-\frac{\theta}{x}}\right]^{\alpha\lambda}\right]^{-h}}_{B} \qquad (11)$$

Using the B term in the equation (11) in relation to equation (7), we have

$$\text{B}=\left[1+\left[e^{-\frac{\theta}{x}}\right]^{\alpha\lambda}\right]^{-h}=\sum_{j=0}^{h}(-1)^j\binom{h+j-1}{j}\left[e^{-\frac{\theta}{x}}\right]^{\alpha\lambda j} \qquad (12)$$

Substituting equations (12) into equation (11), we have

$$\left[F(x,\alpha,\lambda,\theta)\right]^h=2^h\sum_{j=0}^{h}(-1)^j\binom{h+j-1}{j}\left[e^{-\frac{\theta}{x}}\right]^{\alpha\lambda(j+h)} \qquad (13)$$

Again the binomial expansion is applied to $\left[e^{-\frac{\theta}{x}}\right]^{\alpha\lambda(j+h)}$ by adding and subtracting 1 so we have

$$1+\left[e^{-\frac{\theta}{x}}\right]^{\alpha\lambda(j+h)}=\sum_{m=0}^{\infty}(-1)^m\binom{\alpha\lambda(j+h)}{m}\left[e^{-\frac{\theta}{x}}\right]^m \qquad (14)$$

$$1-\left[e^{-\frac{\theta}{x}}\right]^{m}=\sum_{p=0}^{\infty}(-1)^{p}\binom{m}{p}\left[e^{-\frac{\theta}{x}}\right]^{p} \quad (15)$$

Substituting equations (14) and equation (15) into equation (13), we have

$$\left[F(x,\alpha,\lambda,\theta)\right]^{h}=2^{h}\sum_{m,p=0}^{\infty}\sum_{j=0}^{h}(-1)^{j+m+p}\binom{h+j-1}{j}\binom{\alpha\lambda(j+h)}{m}\binom{m}{p}\left[e^{-\frac{\theta}{x}}\right]^{p} \quad (16)$$

The expression above can be rewrite, so we have

$$\left[F(x,\alpha,\lambda,\theta)\right]^{h}=\sum_{j=0}^{h}\delta\left[e^{-\frac{\theta}{x}}\right]^{p} \quad (17)$$

where $\delta=2^{h}\sum_{m,p=0}^{\infty}(-1)^{j+m+p}\binom{m}{p}\binom{h+j-1}{j}\binom{\alpha\lambda(j+h)}{m}$

Now (17) is the important representation of the cdf of TIIHLEtIEx distribution.

2.4 Statistical Properties of TIIHLEtIEx Distribution

This section explores various statistical properties of TIIHLEtIEx distribution.

2.4.1 Probability Weighted Moments of the TIIHLEtIEx Distribution

$$\tau_{r,s}=E\left[X^{r}F(X)^{s}\right]=\int_{-\infty}^{\infty}x^{r}f(x)(F(x))^{s}dx \quad (18)$$

The PWMs of TIIHLEtIEx is derive by substituting equation (9) and equation (16) into equation (17) replacing h with s, and Consider the integral part we have

$$\tau_{r,s}=\sum_{i=0}^{\infty}\sum_{j=0}^{s}\delta\psi\left(\theta\alpha\lambda\right)^{r}\left((i+1)+p\right)^{r-1}\Gamma(1-r) \quad (19)$$

Now $\psi=2(-1)^{i}\binom{i+1}{i}$ and $\delta=2^{s}\sum_{m,p=0}^{\infty}(-1)^{j+m+p}\binom{m}{p}\binom{s+j-1}{j}\binom{\lambda\alpha(j+s)}{m}$

The equation (19) above is the PWMs of TIIHLEtIEx.

2.4.2 Moments of the TIIHLEtIEx Distribution

$$E(X^{r})=\int_{0}^{\infty}x^{r}f(x)dx \quad (20)$$

The r^{th} moments for TIIHLEtIEx distribution is derive by substituting equation (10) into equation (20) we obtain

$$E(X^{r})=\sum_{i,0}^{\infty}\psi\frac{\theta}{x^{2}}\int_{0}^{\infty}x^{r}\left[e^{-\frac{\theta}{x}}\right]^{\alpha\lambda(i+1)}dx \quad (21)$$

Consider the integral part of equation (21)

$$E\left[X^{r}\right]=\sum_{i=0}^{\infty}\psi\left(\theta\alpha\lambda\right)^{r}(i+1)^{r-1}\Gamma(1-r) \quad (22)$$

Now, $\psi=2(-1)^{i}\binom{i+1}{i}$

The equation (22) above is the r^{th} moments for TIIHLEtIEx distribution and the mean of the distribution will be obtained by setting r = 1 in (22).

2.4.3 Moment Generating Function (mgf) of TIIHLEtIEx Distribution

The Moment Generating Function of x is given as

$$M_{x}(t)=\int_{0}^{\infty}e^{tx}f(x)dx \quad (23)$$

The mgf for TIIHLEtIEx distribution is derive by substituting equation (10) into equation (23) we obtain:

$$M_{x}(t)=\sum_{i=0}^{\infty}\delta\frac{\theta}{x^{2}}\int_{0}^{\infty}e^{tx}\left[e^{-\frac{\theta}{x}}\right]^{\alpha\lambda(i+1)}dx \quad (24)$$

where the expansion of $e^{tx}=\sum_{m=0}^{\infty}\frac{t^{m}x^{m}}{m!}$ and following the process of moments above, we have the mgf for TIHLEtIEx distribution in equation (25) below.

$$M_{x}(t)=\sum_{i=0}^{\infty}\sum_{m=0}^{\infty}\delta\frac{t^{m}}{m!}\left(\theta\alpha\lambda\right)^{m}(i+1)^{m-1}\Gamma(1-m) \quad (25)$$

2.4.4 Reliability function of TIIHLEtIEx Distribution

The reliability function is also known as survival function, which is the probability of an item not failing prior to some time. It can be defined as

$$R(x;\lambda,\alpha,\theta)=1-F(x;\lambda,\alpha,\theta) \quad (26)$$

$$R(x;\lambda,\alpha,\theta)=\frac{1-\left[e^{-\frac{\theta}{x}}\right]^{\alpha\lambda}}{1+\left[e^{-\frac{\theta}{x}}\right]^{\alpha\lambda}} \quad (27)$$

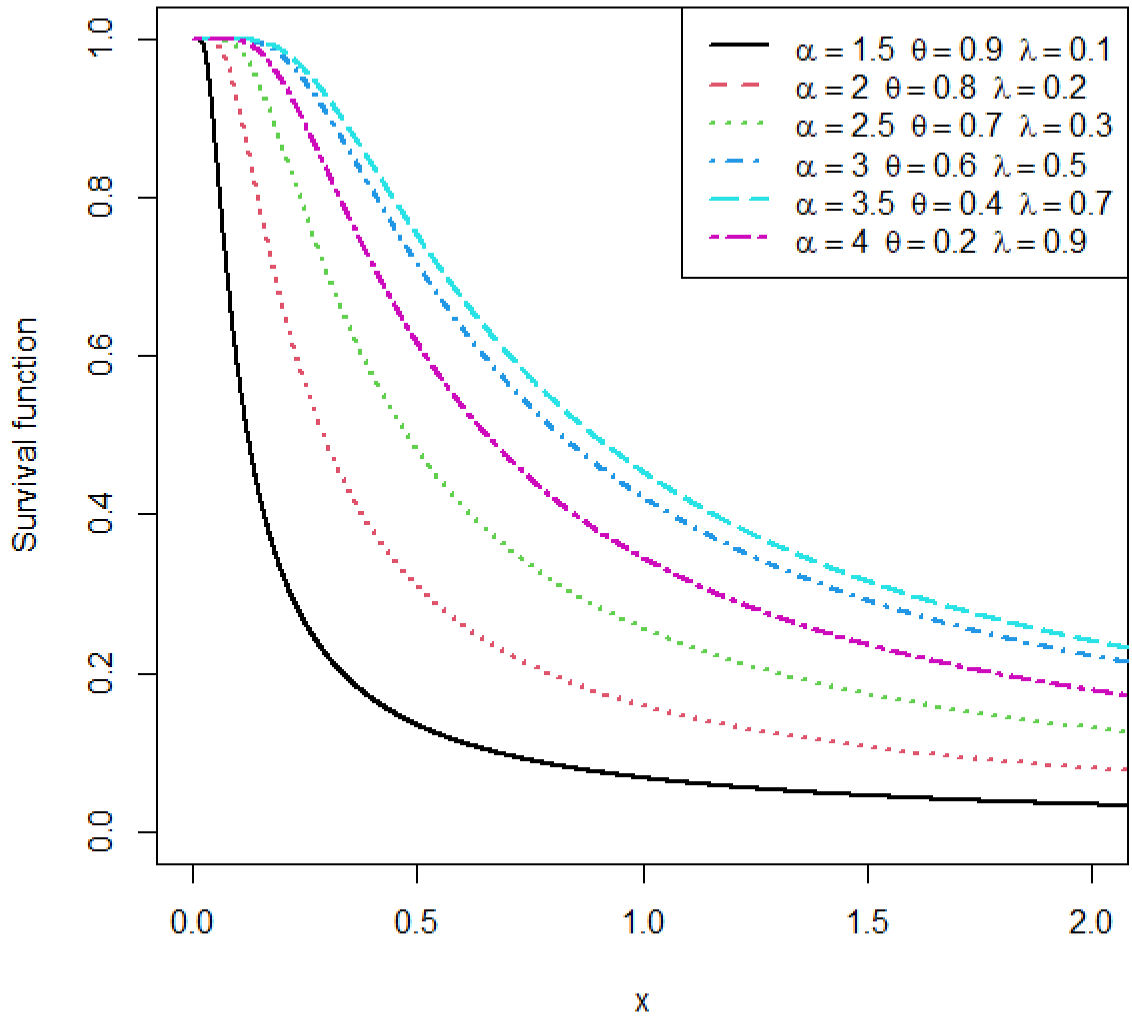

Figure 1: Plots of survival function of TIIHLEtIEx distribution

2.4.5 Hazard Function of TIIHLEtIEx Distribution

The hazard function provides information about how the risk of an event changes over time. It characterizes the failure rate or the rate of occurrence of the event at a particular time, conditional on the event not occurring before that time. It can be defined as

$$T(x;\lambda,\alpha,\theta)=\frac{f(x;\lambda,\alpha,\theta)}{R(x;\lambda,\alpha,\theta)} \quad (28)$$

$$T(x;\lambda,\alpha,\theta)=\frac{2\alpha\lambda\frac{\theta}{x^{2}}e^{-\frac{\theta}{x}}\left[e^{-\frac{\theta}{x}}\right]^{\alpha-1}\left[e^{-\frac{\theta}{x}}\right]^{\alpha(\lambda-1)}}{1-\left[e^{-\frac{\theta}{x}}\right]^{2\alpha\lambda}} \quad (29)$$

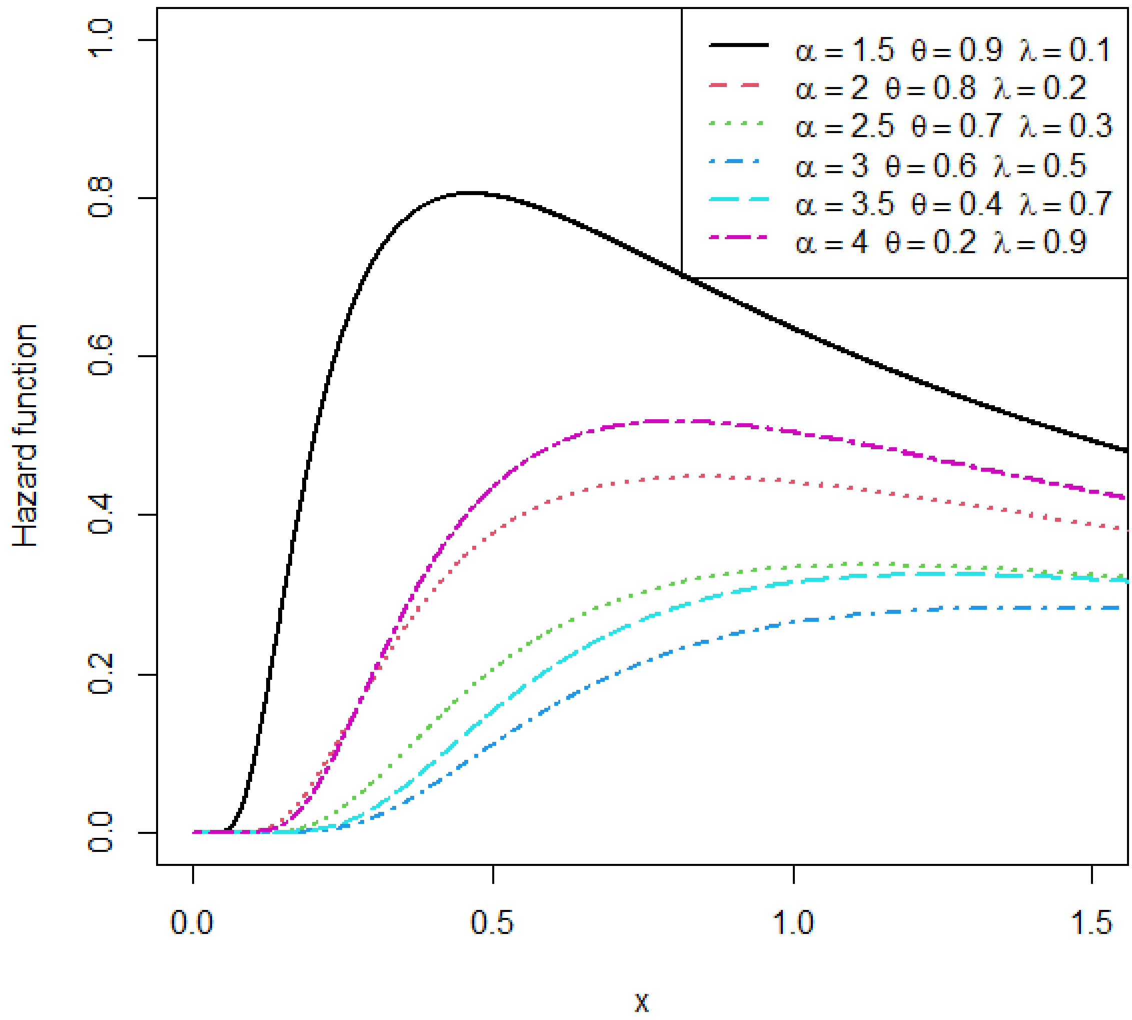

Figure 2: Plots of hazard function of TIIHLEtIEx distribution for different parameter values

Figure 2 shows the shape of the hazard function of TIIHLEtIEx distribution. It can be seen from the plots that it increases gradually, gets to the peak and started decreasing gradually. This is a perfect shape of inverted bathtub.

2.4.6 Quantile Function of TIIHLEtIEx Distribution

The quantile function which is also known as inverse CDF, of the TIIHLEtIEx distribution is obtained by using the CDF in equation (5).

$$F(x;\lambda,\alpha,\theta)=\frac{2\left[e^{-\left(\frac{\theta}{x}\right)}\right]^{\alpha\lambda}}{1+\left[e^{-\left(\frac{\theta}{x}\right)}\right]^{\alpha\lambda}}=U$$

$$2\left[e^{-\left(\frac{\theta}{x}\right)}\right]^{\alpha\lambda}=U\left[1+\left[e^{-\left(\frac{\theta}{x}\right)}\right]^{\alpha\lambda}\right]$$

$$x=Q(u)=\frac{\theta}{\left[-\log\left[\frac{U}{2-U}\right]^{\frac{1}{\alpha\lambda}}\right]} \tag{30}$$

The median of the TIIHLEtIEx distribution can be derived by substituting u = 0:5 in (38) as follows:

$$\text{median}=Q(0.5)=\frac{\theta}{\left[-\log\left[\frac{0.5}{2-0.5}\right]^{\frac{1}{\alpha\lambda}}\right]} \tag{31}$$

2.5 Distribution of Order Statistics of TIIHLEtIEx Distribution

Let $X_1, X_2,..., X_3$ be independent and identically distributed (i.i.d) random variables with their corresponding continuous distribution function F(x). Let $X_{1:n}<X_{2:n}<...<X_{n:n}$ the corresponding ordered random sample from the TIIHLEtIEx distributions. Let $F_{r:n}(x)$ and $f_{r:n}(x)$, r=1,2,3,...,n denote the CDF and PDF of the r^{th} order statistics $X_{r:n}$ respectively. The PDF of the r^{th} order statistics of $X_{r:n}$ is given as

$$f_{r:n}(x;\lambda,\alpha,\theta)=\frac{f(x)}{B(r,n-r+1)}\sum_{v=0}^{n-r}(-1)^{v}\binom{n-r}{v}\left[F(x)\right]^{v+r-1} \quad (32)$$

The PDF of r^{th} order statistic for TIIHLEtIEx distribution is derived by substituting equation (10) and equation (17) into equation (32). Also replacing h with v+r-1 in equation (17), so we have

$$f_{r:n}(x;\lambda,\alpha,\theta)=\frac{1}{B(r,n-r+1)}\sum_{v=0}^{n-r}\sum_{i=0}^{\infty}\sum_{j=0}^{r+v-1}(-1)^{v}\binom{n-r}{v}\delta\psi\left[e^{-\frac{\theta}{x}}\right]^{\alpha\lambda(i+1)+p} \quad (33)$$

Now

$$\delta=2^{v+r-1}\sum_{m,p=0}^{\infty}(-1)^{j+m+p}\binom{\mathrm{m}}{p}\binom{v+r+j-2}{j}\binom{\alpha\lambda\left(j+(v+r-1)\right)}{m}$$

and

$$\psi=2\alpha\lambda\frac{\theta}{x^{2}}(-1)^{i}\binom{i+1}{i}$$

The PDF of minimum order statistic of the TIIHLEtIEx distribution is obtained by setting r = 1 in equation (33) as

$$f_{1:n}(x;\lambda,\alpha,\theta)=2^{v+1}\alpha\lambda n\frac{\theta}{x^{2}}\sum_{v}^{n-1}\sum_{i=0}^{\infty}\sum_{m,p=0}^{\infty}\sum_{j=0}^{v}(-1)^{v+i+j+m+p}\binom{i+1}{i}\binom{m}{p}\binom{v+j-1}{j}\binom{\alpha\lambda\left(j+v\right)}{m}\left[e^{-\frac{\theta}{x}}\right]^{\alpha\lambda(i+1)+p} \quad (34)$$

Also, the PDF of maximum order statistic of the TIIHLEtIEx distribution is obtained by setting r = n in equation (34) as

$$f_{n:n}(x;\lambda,\alpha,\theta)=2^{v+n}\alpha\lambda n\frac{\theta}{x^{2}}\sum_{i=0}^{\infty}\sum_{m,p=0}^{\infty}\sum_{j=0}^{n+v-1}(-1)^{v+i+j+m+p}\binom{i+1}{i}\binom{m}{p}\binom{v+n+j-2}{j}\binom{\alpha\lambda\left(j+(v+n-1)\right)}{m}\left[e^{-\frac{\theta}{x}}\right]^{\alpha\lambda(i+1)+p} \quad (35)$$

2.6 Parameter Estimation of TIIHLEtIEx Distribution

2.6.1 Maximum Likelihood Estimation of TIIHLEtIEx Distribution

Let $x_1, x_2, x_3,..., x_n$ be a random sample of size n from the TIIHLEtIEx distribution. Then, the likelihood function based on observed sample for the vector of parameter $(\lambda,\alpha,\theta)^T$ is given

$$\log(L)=n\log(2)+n\log(\alpha)+n\log(\lambda)+n\log(\theta)+\sum_{i=1}^{n}\log\left(\frac{1}{x_i^{2}}\right)-\sum_{i=1}^{n}\left(\frac{\theta}{x_i}\right)^{\alpha}-\sum_{i=1}^{n}\left(\frac{\theta}{x_i}\right)^{\alpha(\lambda-1)}-2\sum_{i=1}^{n}\log\left[1+\left[e^{-\frac{\theta}{x_i}}\right]^{\alpha\lambda}\right] \quad (36)$$

Differentiating the log-likelihood with respect to λ,α,θ and equating the result to zero, we have

$$\frac{\partial L}{\partial\alpha}=\frac{n}{\alpha}-\sum_{i=1}^{n}\left(\frac{\theta}{x_i}\right)^{\alpha}\log\left(\frac{\theta}{x_i}\right)-\sum_{i=1}^{n}\left(\frac{\theta}{x_i}\right)^{\alpha(\lambda-1)}\log\left(\frac{\theta}{x_i}\right)^{(\lambda-1)}-2\lambda\sum_{i=1}^{n}\frac{\left[e^{-\frac{\theta}{x_i}}\right]^{\alpha(\lambda-1)}\left[e^{-\frac{\theta}{x_i}}\right]^{\alpha}\log\left[e^{-\frac{\theta}{x_i}}\right]}{1+\left[e^{-\frac{\theta}{x_i}}\right]^{\alpha\lambda}}=0 \quad (37)$$

$$\frac{\partial L}{\partial \lambda}=\frac{n}{\lambda}+\alpha\sum_{i=1}^{n}\left(\frac{\theta}{x_i}\right)^{\alpha(\lambda-1)}\log\left(\frac{\theta}{x_i}\right)-2\sum_{i=1}^{n}\frac{\left[e^{-\frac{\theta}{x_i}}\right]^{\alpha\lambda}\log\left[e^{-\frac{\theta}{x_i}}\right]^{\alpha}}{1+\left[e^{-\frac{\theta}{x_i}}\right]^{\alpha\lambda}}=0 \quad (38)$$

$$\frac{\partial L}{\partial \theta}=\frac{n}{\theta}-\alpha\sum_{i=1}^{n}\left(\frac{\theta}{x_i}\right)^{\alpha-1}\frac{1}{x_i}+\alpha(\lambda-1)\sum_{i=1}^{n}\left(\frac{\theta}{x_i}\right)^{\alpha(\lambda-1)-1}\frac{1}{x_i}$$

$$+2\alpha\lambda\sum_{i=1}^{n}\frac{\left[e^{-\frac{\theta}{x_i}}\right]^{\alpha-1}\left[e^{-\frac{\theta}{x_i}}\right]^{\alpha(\lambda-1)}e^{-\frac{\theta}{x_i}}\frac{1}{x_i}}{1+\left[e^{-\frac{\theta}{x_i}}\right]^{\alpha\lambda}}=0 \quad (39)$$

The equation (37), (38) and (39) above are non linear, cannot be solved analytically, necessitating the use of analytical tools to solve them numerically.

IV. RESULTS AND DISCUSSION

3.1 Simulation Study of Type II Half-Logistic Exponentiated Inverse Exponential (TIIHLEtIEx) Distribution

In this study, we generated 1000 replicates from the TIIHLEtIEx distribution using the quantile function described in equation (38). We selected sample sizes of n = 20, 50, 100, 150, 200, and 250 for the analysis. The resulting replicates were used to calculate parameter estimates, bias, and Root Mean Square Error (RMSE), which are presented in Tables 3.1. These table show the Maximum Likelihood Estimate (MLE) estimates, along with the corresponding bias and RMSE for specific values of λ = 1, α =1, θ =3, and λ = 0.5, α =0.5, θ =1.2 for the parameters. The findings from the table 1 indicate that as the sample size increases, the biases and RMSEs approach zero, implying that the estimates become more accurate and reliable. This suggests that the estimates obtained are both efficient and consistent, as larger sample sizes lead to better parameter estimation. .

Table 3.1: MLEs, biases and RMSE for some values of parameters

		(1,1,3)			(0.5,0.5,1.2)		
N	Parameters	Estimated Values	Bais	RMSE	Estimated Values	Bais	RMSE
20	λ α θ	0.9301 1.0641 3.1358	-0.0699 0.0641 0.1358	0.1585 0.1346 0.1563	0.7819 1.1085 1.4314	0.2819 0.6085 0.2314	0.3001 0.6268 0.2346
50	λ α θ	0.9289 1.0470 3.1323	-0.0711 0.0470 0.1323	0.1371 0.0992 0.1505	0.7737 1.1003 1.4288	0.2737 0.6003 0.2288	0.2852 0.6103 0.2307
100	λ α θ	0.9054 1.0418 3.1251	-0.0946 0.0418 0.1251	0.1237 0.0987 0.1429	0.7646 1.0001 1.4102	0.2646 0.0001 0.2102	0.2713 0.5150 0.1319
150	λ α θ	1.0933 1.0396 3.1240	0.0933 0.0396 0.1240	0.1139 0.0925 0.1414	0.5546 0.5356 1.3298	0.0546 0.0356 0.1298	0.1627 0.5048 0.1314
200	λ α θ	1.0368 1.0358 3.1217	0.0368 0.0358 0.1217	0.1080 0.0910 0.1409	0.5405 1.0998 1.2268	0.0405 0.5998 0.0268	0.1314 0.3055 0.1240

250	λ α θ	1.0211 1.0339 3.1124	0.0211 0.0339 0.1124	0.0994 0.0866 0.1346	0.5110 1.0481 1.2085	0.0110 0.5481 0.0085	0.1143 0.3046 0.1137

3.2 Application of the new models to Real-Life Data sets

Data Set 1

The first data set shown below represents the strength of carbon fibers tested under tension at gauge lengths of 10mm, previously used Bi and Gui (2017):

1.901, 2.132, 2.203, 2.228, 2.257, 2.350, 2.361, 2.396, 2.397, 2.445, 2.454, 2.474, 2.518, 2.522, 2.525, 2.532, 2.575, 2.614, 2.616, 2.618, 2.624, 2.659, 2.675, 2.738, 2.740, 2.856, 2.917, 2.928, 2.937, 2.937, 2.977, 2.996, 3.030, 3.125, 3.139, 3.145, 3.220, 3.223, 3.235, 3.243, 3.264, 3.272, 3.294, 3.332, 3.346, 3.377, 3.408, 3.435, 3.493, 3.501, 3.537, 3.554, 3.562, 3.628, 3.852, 3.871, 3.886, 3.971, 4.024, 4.027, 4.225, 4.395, 5.020

Data Set 2

The second data set shown below represents the mortality rate of the COVID-19 patients in Canada, previously used by Liu et al., (2021):

3.1091, 3.3825, 3.1444, 3.2135, 2.4946, 3.5146, 4.9274, 3.3769, 6.8686, 3.0914, 4.9378, 3.1091, 3.2823, 3.8594, 4.0480, 4.1685, 3.6426, 3.2110, 2.8636, 3.2218, 2.9078, 3.6346, 2.7957, 4.2781, 4.2202, 1.5157, 2.6029, 3.3592, 2.8349, 3.1348, 2.5261, 1.5806, 2.7704, 2.1901, 2.4141, 1.9048.

Fitting Type II Half-Logistic Exponentiated Inverse Exponential Distribution

In this section, real-life applications of two data sets are discussed. To illustrate the effectiveness of a new distribution, a comparative study is conducted by fitting the Type II Topp-Leone Inverse Exponential (TIIToLIEx), Exponentiated Inverse Exponential (EIEx), Inverse Exponential (IEx), and Kumaraswamy Inverse Exponential (KIEx) distributions to the data. The analysis is carried out using the Adequacy Model package in the R software.

3.3 The Comparators

The pdf of the comparators considered are:

- Topp-Leone Inverse Exponential (TIIToLIEx)Distribution

$$f\left(x;\theta,\beta\right)=2\theta\alpha x^{-2}e^{-\frac{2\alpha}{x}}\left[1-e^{-\frac{2\alpha}{x}}\right]^{\theta-1}$$

- Exponentiated Inverse Exponential (EIEx) Distribution

$$f\left(x;\alpha,\beta\right)=\frac{\alpha\beta}{x^2}\left(e^{-\frac{\beta}{x}}\right)^{\alpha}$$

- Inverse Exponential (IEx) Distribution

$$f\left(x;\beta\right)=\frac{\beta}{x^2}e^{-\frac{\beta}{x}}$$

- Kumaraswamy Inverse Exponential (KIEx) Distribution

$$f\left(x;\alpha,\lambda,\beta\right)=\alpha\lambda\left(\frac{\beta}{x^2}\right)e^{-\frac{\alpha\beta}{x}}\left[1-e^{-\frac{\alpha\beta}{x}}\right]^{\lambda-1}$$

3.4 Results with Comparators

In this section, a comparative analysis is performed using the baseline inverse exponential distribution as a reference point. The main objective is to evaluate the impact of introducing additional parameters to the distribution and how this affects its flexibility, applicability, and overall effectiveness. By comparing the baseline distribution with the new distributions that have additional parameters, we can gain insights into how these modifications enhance or alter the distribution's ability to fit real-life data and address various modeling challenges.

Table 3.2 The MLEs, Log-likelihoods and Goodness of Fits Statistics of the models based on the strength of carbon fibers tested under tension at gauge lengths of 10mm (Data set 3)

Model	α	θ	λ	β	LL	AIC
TIIHLEtIEx	0.4917	3.9449	2.3233	-	-57.7252	121.4504
TIIToLIEx	6.6649	0.0519	-	-	- 60.3957	124.7914
EIEx	1.8299	-	-	1.6077	-133.4229	270.8458
Iex	-	-	-	2.9424	-133.4229	268.8458
KIEx	0.1998	-	4.1397	5.3261	-121.8617	249.7234

Table 3.2 presents the results of the Maximum Likelihood Estimation for the parameters of the TIIHLEtIEx distribution and four other comparable distributions. After analyzing the goodness of fit measure, it was found that the TIIHLEtIEx distribution obtained the lowest AIC value of 121.4504. As a consequence, out of all the distributions considered, the TIIHLEtIEx distribution showed the most favorable fit to the strength of carbon fibers tested under tension at gauge lengths of 10mm. This suggests that the TIIHLEtIEx distribution is the most suitable model for describing the strength of carbon fibers tested under tension at gauge lengths of 10mm compared to the other distributions tested.

Table 3.3 The MLEs, Log-likelihoods and Goodness of Fits Statistics of the models based on daily confirmed cases of COVID-19 (Data set 4)

Model	α	θ	λ	β	LL	AIC
TIIHLEtIEx	1.1754	1.7480	2.2249	-	-48.0901	102.1802
TIIToLIEx	6.1478	3.2197	-	-	-50.2501	104.5002
EIEx	1.8176	-	-	1.6551	-78.8355	161.6711
Iex	-	-	-	3.0084	-78.8355	159.6711
KIEx	9.6424	-	0.6259	3.2066	-72.8357	151.6714

Table 3.3 presents the results of the Maximum Likelihood Estimation for the parameters of the TIIHLEtIEx distribution and four other comparable distributions. After assessing the goodness of fit measure, it was found that the TIIHLEtIEx distribution achieved the lowest AIC value of 102.1802. Consequently, out of all the distributions considered, the TIIHLEtIEx distribution exhibited the best fit to the daily confirmed COVID-19 dataset. This indicates that the TIIHLEtIEx distribution is the most suitable model for describing the COVID-19 data compared to the other distributions that were examined.

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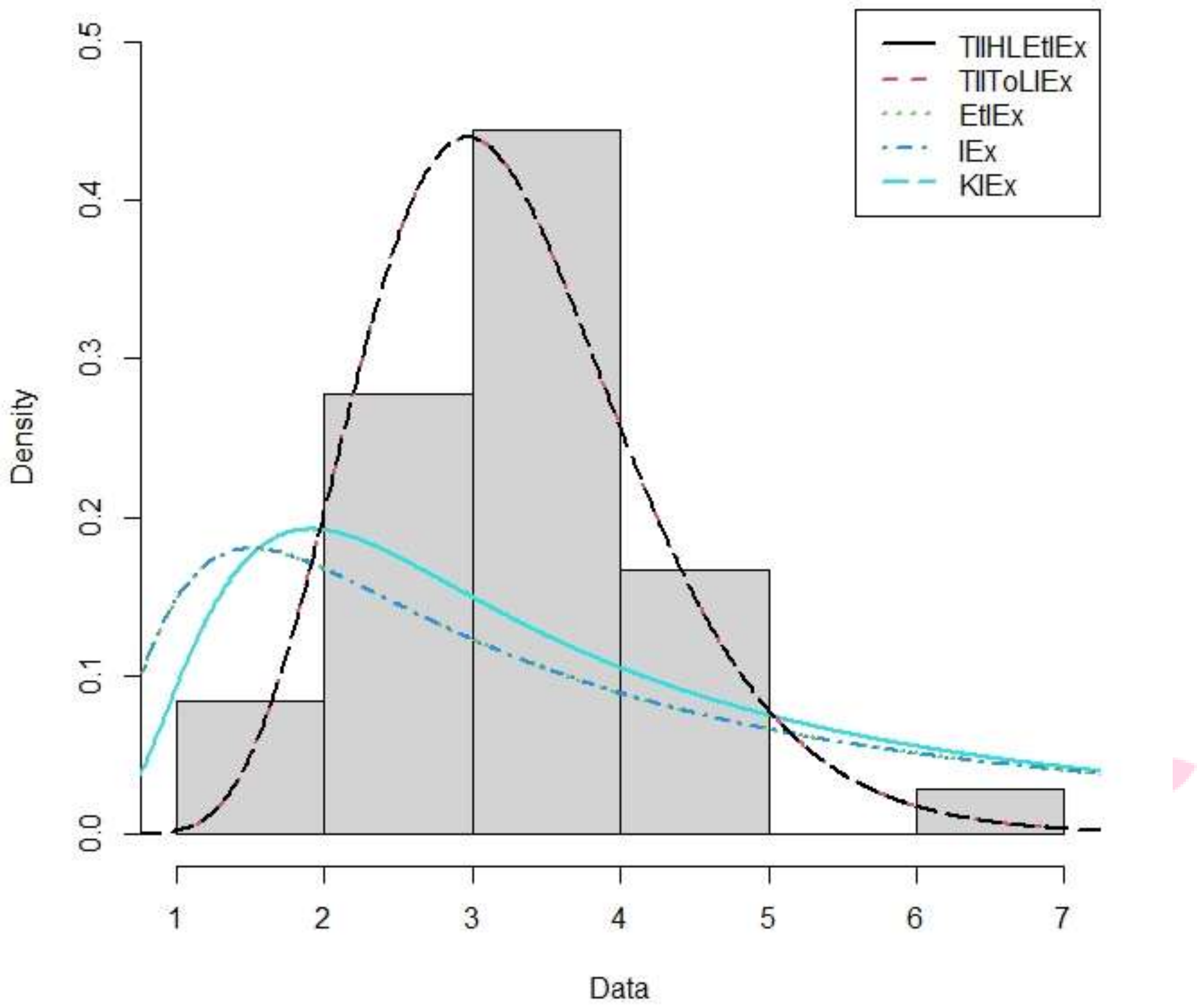

Figure 1: Fitted pdfs for the TIIHLEtIEx, TIIToLIEx, EIEx, IEx, and KIEx models to the data set 1.

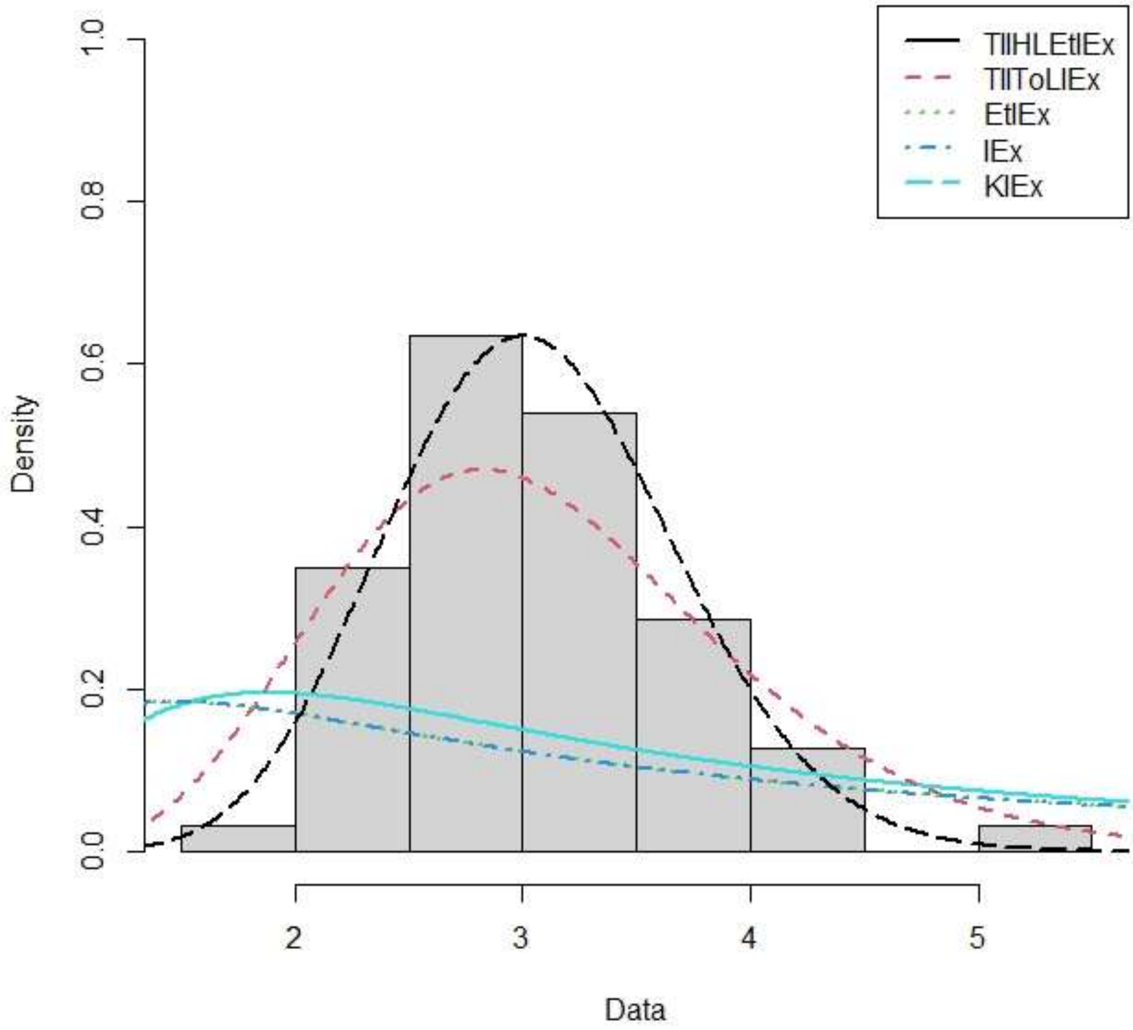

Figure 2: Fitted pdfs for the TIIHLEtIEx, TIIToLIEx, EIEx, IEx, and KIEx models to the data set 2.

Figures 1 and 2 present visual representations of the fit of the TIIHLEtIEx distribution and its comparator distributions. By examining these plots, it becomes clear that the TIIHLEtIEx distribution exhibits better performance in fitting the data compared to the other distributions. This visual evidence further reinforces the conclusion that the TIIHLEtIEx distribution surpasses its comparator distributions in accurately describing the dataset under consideration.

I. Acknowledgment

In this paper, a comparative analysis is performed using the baseline inverse exponential distribution as a reference point. The main objective is to evaluate the impact of introducing additional parameters to the distribution and how this affects its flexibility, applicability, and overall effectiveness. By comparing the baseline distribution with the new distributions that have additional parameters, we can gain insights into how these modifications enhance or alter the distribution's ability to fit real-life data and address various modeling challenges.

II. The results of AIC showed that the TIIHLEtIEx distribution obtained the lowest AIC value as compared with others comparators distributions on the applications to real-life data sets.. Also, the result of the goodness of fit showed that TIIHLEtIEx distribution is the most favorable fit to the strength of carbon fibers tested under tension at gauge lengths of 10mm and COVID-19 data.

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