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The exact solutions for the fractional reaction-diffusion equations

Dr. Amrita Singh
Department of Mathematics, (Guest Faculty) C. M. Science College Darbhanga,
L.N.M University, Darbhanga

Abstract: The prime objective of this paper is to obtain some new families of exact solitary wave solutions of the nonlinear partial differential equations (NPDEs) via computerised symbolic computation on *Wolfram Mathematica*. By applying the sine-Gordon expansion method, numerous exact soliton solutions are constructed for the NPDEs, which provide a model of the interaction between the Langmuir wave and the ion-acoustic wave in high-frequency plasma. Consequently, the exact solitary wave solutions are obtained in different forms of dynamical wave structures of solitons including multi-solitons, lump-type solitons, travelling waves, kink waves, also trigonometric and hyperbolic function solutions, and rational function solutions. Moreover, the dynamical behaviour of the resulting multiple soliton solutions is discussed both analytically and graphically by using suitable values of free parameters through numerical simulation. The reported results have rich physical structures that are helpful to explain the nonlinear wave phenomena in plasma physics and soliton theory.

Keywords: Reaction-Diffusion Equation; Exact Solutions; Conformal Derivative; Sine-Gordon Expansion Method (SGEM).

1. Introduction

Nonlinear partial differential equations (PDEs) are mathematical equations that involve partial derivatives of an unknown function with respect to two or more independent variables and the function itself, and where the relationship between the variables is nonlinear. These equations are widely used to model complex physical phenomena in various fields, including physics, engineering, biology, and finance. Unlike linear PDEs, where the unknown function and its derivatives appear linearly, nonlinear PDEs can exhibit more intricate and often more challenging behaviour.

A general form of a nonlinear PDE can be expressed as:

$$F\left(x, t, u, \frac{\partial u}{\partial x}, \frac{\delta u}{\delta t}, \frac{\partial^2 u}{\partial x^2}, \dots\right) = 0 \qquad (1)$$

Here, u is the unknown function, and F is a nonlinear expression involving u and its partial derivatives. The specific form of F depends on the physical or mathematical context of the problem being modelled [1, 2, 3, 4, 5].

Solving nonlinear PDEs analytically is often difficult or impossible, and numerical methods are frequently employed for obtaining approximate solutions. Some commonly used numerical methods for solving nonlinear PDEs include finite difference methods, finite element methods, and spectral methods [6, 7, 8].

Nonlinear PDEs arise in various areas of science and engineering. Examples include the nonlinear Schrödinger equation in quantum mechanics, the Navier-Stokes equations in fluid dynamics, the reaction-diffusion equations in chemistry and biology, and the nonlinear heat equation in heat conduction with temperature-dependent properties [9, 10, 11].

The study of nonlinear PDEs is a rich and active area of research, and researchers develop both theoretical and numerical techniques to understand and solve these equations in various applications.

Let's consider fractional reaction-diffusion equations given by as follows: [12]

$$u_{tt}^{\theta} + \alpha u_{xx} + \beta u + \gamma u^3 = 0,\ 0 < \theta \leq 1. \qquad (2)$$

Here α, β and γ are arbitrary constant.

In this study, we aim of submit for soliton structures of Eq. (2) by using the sine– Gordon expansion method. This paper has been organised as follows: In Sect.2, we present the few basic definition conformal derivative. In Sect. 3, we present the description the method. In Sect. 4, SGEM has been applied to the reaction-diffusion equations. Finally, in Sect. 5, we presented the conclusions of the study comprehensively

2. Definitions

2.1 Let f: $[a, b] \to \mathbb{R}$ be a continuous nonnegative map such that $f(t) \neq 0$, whenever t $>$ a. Given a function h: $[a, b] \to \mathbb{R}$ and $\boldsymbol{\alpha} \in (0, 1)$ a real, we say that h is $\boldsymbol{\alpha}$ – differentiable at t $>$ a, with respect to kernel f, if the limit

$$\frac{\partial^{\alpha}}{\partial t^{\alpha}} h(\mathrm{t}) := \lim_{\varepsilon \to 0} \frac{h\left(t+\varepsilon f(t)^{1-\alpha}\right) - h(t)}{\varepsilon} \qquad (3)$$

exists. The $\alpha -$ derivative at t = a is defined by

$$\frac{\partial^{\alpha}}{\partial t^{\alpha}} h(a) := \lim_{t \to a^{+}} \frac{\partial^{\alpha}}{\partial t^{\alpha}} h(t) \qquad (4)$$

if the limit exists.

Consider the limit $\alpha \to 1^{-}$. In this case, for $t > a$, is obtained the classical definition for derivative of a function, $\frac{\partial^{\alpha}}{\partial t^{\alpha}} h(t) = \frac{d}{dt} h(t)$.

The following result shows the relation with the usual derivative of integer order.

2.2 Let h: $[a, b] \to \mathbb{R}$ be a differentiable function and t $>$ a. Then, f is α – differentiable at t and

$$\frac{\partial^{\alpha}}{\partial \mathrm{t}^{\alpha}} h(t) = f(t)^{1-\alpha} \frac{d}{dt} h(t), \qquad \mathrm{t} > \mathrm{a} \qquad (5)$$

Also, if $\frac{dh}{dt}$ is continuous at t = a, then

$$\frac{\partial^{\alpha}}{\partial t^{\alpha}} h(a) = f(a)^{1-\alpha} \frac{\mathrm{d}}{\mathrm{dt}} h(a) \qquad (6)$$

However, there exist α – differentiable functions which are not differentiable in the usual sense.

The definition in 2.1 satisfies the properties given following:

2.3 Let α ∈ (0,1] and h, g be α – differentiable at point t > 0, then:

a) $\frac{\partial^\alpha}{\partial t^\alpha}(ch+dg) = c\frac{\partial^\alpha}{\partial t^\alpha}(h) + d\frac{\partial^\alpha}{\partial t^\alpha}(g)$, for all a, b ∈ ℝ
b) $\frac{\partial^\alpha}{\partial t^\alpha}(t^p) = f(t)^{1-\alpha}pt^{p-1}$ for all p ∈ ℝ
c) $\frac{\partial^\alpha}{\partial t^\alpha}(\lambda) = 0$ for all constant function $h(t) = \lambda$
d) $\frac{\partial^\alpha}{\partial t^\alpha}(hg) = h\frac{\partial^\alpha}{\partial t^\alpha}(g) + g\frac{\partial^\alpha}{\partial t^\alpha}(h)$
e) $\frac{\partial^\alpha}{\partial t^\alpha}\left(\frac{h}{g}\right) = \frac{g\frac{\partial^\alpha}{\partial t^\alpha}(h)-h\frac{\partial^\alpha}{\partial t^\alpha}(g)}{g^2}$

3. General Facts of the Sine-Gordon expansion method

In this section we discuss the general facts of SGEM. Consider the following Sine-Gordon equation

$$u_{xx} - u_{tt} = m^2\sin(u), \tag{7}$$

Where $u = u(x,t)$ and m is a real constant.

Applying the wave transformation $u = u(x,t) = U(\xi)$, $\xi = \mu(x-ct)$ to Eq. (7), yields the following nonlinear ordinary differential equation (NODE):

$$U'' = \frac{m^2}{\mu^2(1-c^2)}\sin(U), \tag{8}$$

Where $U = U(\xi)$, ξ is the amplitude of the travelling wave and c is the velocity of the travelling wave. Reconsidering Eq. (8), we can write its full simplification as:

$$\left[\left(\frac{U}{2}\right)'\right] = \frac{m^2}{\mu^2(1-c^2)}sin^2\left(\frac{U}{2}\right) + K, \tag{9}$$

Where K is the integration constant.

Substituting K = 0, $w(\xi) = \frac{U}{2}$ and $a^2 = \frac{m^2}{\mu^2(1-c^2)}$ in Eq. (9), gives:

$$w' = a\sin(w), \tag{10}$$

Putting a = 1 in Eq. (10), we have:

$$w' = \sin(w), \tag{11}$$

Equation (11) is variables separable equation, we obtain the following two significant equations from solving it:

$$\sin(w) = \sin(w(\xi)) = \frac{2pe^\xi}{p^2e^{2\xi}+1}\Big|_{p=1} = \sec h(\xi), \tag{12}$$

$$\cos(w) = \cos(w(\xi)) = \frac{p^2e^{2\xi}-1}{p^2e^{2\xi}+1}\Big|_{p=1} = \tan h(\xi), \tag{13}$$

Where p is the integral constant.

For the solution of the following nonlinear partial differential equation;

$$P(u, u_x, u_t, u_{xx}, u_{tt}, u_{xt} \ldots), \tag{14}$$

We consider,

$$U(\xi) = \sum_{i=1}^{n} \tan h^{i-1}(\xi)[B_i \sec h(\xi) + A_i \tan h(\xi)] + A_0, \tag{15}$$

Equation (15) can be rewritten according to Eqs. (13) and (14) as follows:

$$U(w) = \sum_{i=1}^{n} cos^{i-1}(w)[B_i \sin(w) + A_i \cos(w)] + A_0. \tag{16}$$

We determine the value n under the terms of NODE by the balance principle. Letting the coefficients of $sin^i(w)cos^j(w)$ to be all zero, yields a system of equations. Solving this system by using Wolfram Mathematica 9 gives the values of A_i, B_i, μ and c. Finally, substituting the values of A_i, B_i, μ and c in Eq. (16), we obtain the new travelling wave solutions to Eq. (15).

4. Application of SGEM to the conformable reactional diffusion equation

In this section, we implement the SGEM to the reaction-diffusion equations. Let's start supposing that the travelling wave transformation as following:

$$u(x,t) = U(\xi) = ax - v\frac{t^{\theta}}{\theta} \tag{17}$$

where a is the wave height and v is the wave velocity. We obtained the following partial derivatives of $U(\xi)$ functions respect to ξ. Substituting Eq. (2) into Eq. (17), the following NODEs are obtained.

$$(v^2 + a^2)U'' + \beta U + \gamma U^3 = 0 \tag{18}$$

Here v, a, β and γ are real constants and non-zero. When we reconsider the Eq. (13) for homogenous balance method between U'' and U^3, the value of $n = 1$ is obtained as following:

$$U(w) = B_1 \sin(w) + A_1 \cos(w) + A_0 \tag{19}$$

Differentiating Eq. (19) twice yields

$$U''(w) = B_1 cos^2(w)\sin(w) - B_1 sin^3(w) - 2A_1 sin^2(w)\cos(w) \tag{20}$$

Putting Eqs. (19) and Eq. (20) into Eq. (18), we get the following trigonometric functions equation:

$$\beta A_0 + \gamma A_0^3 + \frac{3}{2}\gamma A_0 A_1^2 + 3\gamma \sin[2w] A_0 A_1 B_1 + \frac{3}{2}\gamma A_0 B_1^2 + \cos[2w]\left(\frac{3}{2}\gamma A_0 A_1^2 - \frac{3}{2}\gamma A_0 B_1^2\right) + \cos[3w]\left(\frac{1}{2}a^2\alpha A_1 + \frac{v^2 A_1}{2} + \frac{\gamma A_1^3}{4} - \frac{3}{4}\gamma A_1 B_1^2\right) + \cos[w]\left(-\frac{1}{2}a^2\alpha A_1 + \beta A_1 - \frac{v^2 A_1}{2} + 3\gamma A_0^2 A_1 + \frac{3\gamma A_1^3}{4} + \frac{3}{4}\gamma A_1 B_1^2\right) + \sin[3w]\left(\frac{1}{2}a^2\alpha B_1 + \frac{v^2 B_1}{2} + \frac{3}{4}\gamma A_1^2 B_1 - \frac{\gamma B_1^3}{4}\right) + \sin[w]\left(-\frac{1}{2}a^2\alpha B_1 + \beta B_1 - \frac{v^2 B_1}{2} + 3\gamma A_0^2 B_1 + \frac{3}{4}\gamma A_1^2 B_1 + \frac{3\gamma B_1^3}{4}\right) \tag{21}$$

Setting each summation of the coefficients of the trigonometric identities of the same power to be zero, yields the following algebraic set of equations.

$$\begin{cases} \text{Constants: } \beta A_0+\gamma A_0^3+\frac{3}{2}\gamma A_0 A_1^2+\frac{3}{2}\gamma A_0 B_1^2 \\ \quad \text{Sin}[2w]: \quad 3\gamma A_0 A_1 B_1 \\ \quad \cos[2w]: \frac{3}{2}\gamma A_0 A_1^2-\frac{3}{2}\gamma A_0 B_1^2 \\ \cos[3w]: \left(\frac{1}{2}a^2\alpha A_1+\frac{v^2 A_1}{2}+\frac{\gamma A_1^3}{4}-\frac{3}{4}\gamma A_1 B_1^2\right) \\ \cos[w]: \frac{v^2 A_1}{2}+3\gamma A_0^2 A_1+\frac{3\gamma A_1^3}{4}+\frac{3}{4}\gamma A_1 B_1^2) \\ \sin[3w]: \left(\frac{1}{2}a^2\alpha B_1+\frac{v^2 B_1}{2}+\frac{3}{4}\gamma A_1^2 B_1-\frac{\gamma B_1^3}{4}\right) \\ \sin[3w]: \left(\frac{1}{2}a^2\alpha B_1+\frac{v^2 B_1}{2}+\frac{3}{4}\gamma A_1^2 B_1-\frac{\gamma B_1^3}{4}\right) \end{cases} \tag{22}$$

By solving the algebraic set of equations, we obtain following values of the coefficients:

Case-1 When we consider following coefficients $A_0=0$, $A_1=\frac{i\sqrt{\beta}}{\sqrt{r}}$, $B_1=\frac{\sqrt{\beta}}{\sqrt{r}}$, $v=-\sqrt{-a^2\alpha+2\beta}$, these produce new combined complex dark- bright soliton solution as:

$$u_1(x,t)=\frac{\sqrt{\beta}\,\text{sech}\left[ax+\frac{t^{\theta}\sqrt{-a^2\alpha+2\beta}}{\theta}\right]}{\sqrt{r}}+\frac{i\sqrt{\beta}\,\tanh\left[ax+\frac{t^{\theta}\sqrt{-a^2\alpha+2\beta}}{\theta}\right]}{\sqrt{r}}. \tag{23}$$

here $r\neq 0$.

Case-2 If it is taken a $A_0=0$, $A_1=-\frac{i\sqrt{\beta}}{\sqrt{r}}$, $B_1=0$, $v=\frac{\sqrt{-2a^2\alpha+\beta}}{\sqrt{2}}$ these produce new dark solution as:

$$u_2(x,t)=\frac{i\sqrt{\beta}\,\tanh\left[ax-\frac{t^{\theta}\sqrt{-2a^2\alpha+\beta}}{\sqrt{2}\theta}\right]}{\sqrt{r}}. \tag{24}$$

here $r\neq 0$.

Case-3 When we take $A_0=0$, $A_1=0$, $B_1=\frac{i\sqrt{2}\sqrt{\beta}}{\sqrt{r}}$, $v=-\sqrt{-a^2\alpha-\beta}$ they produce a new singular solution as

$$u_3(x,t)=\frac{i\sqrt{2}\sqrt{\beta}\,\text{sech}\left[ax+\frac{t^{\theta}\sqrt{-a^2\alpha-\beta}}{\theta}\right]}{\sqrt{r}}. \tag{25}$$

here $r\neq 0$.

Case-4 When we take $A_0=0$, $A_1=-\frac{i\sqrt{\beta}}{\sqrt{r}}$, $B_1=-\frac{\sqrt{\beta}}{\sqrt{r}}$, $v=\sqrt{-a^2\alpha+2\beta}$ give mixed complex conjugate dark-bright solution as:

$$u_4(x,t) = -\frac{\sqrt{\beta}\,\mathrm{sech}\left[ax - \frac{t^{\theta}\sqrt{-a^2\alpha+2\beta}}{\theta}\right]}{\sqrt{r}} - \frac{i\sqrt{\beta}\tanh\left[ax - \frac{t^{\theta}\sqrt{-a^2\alpha+2\beta}}{\theta}\right]}{\sqrt{r}}. \quad (25)$$

here $r \neq 0$.

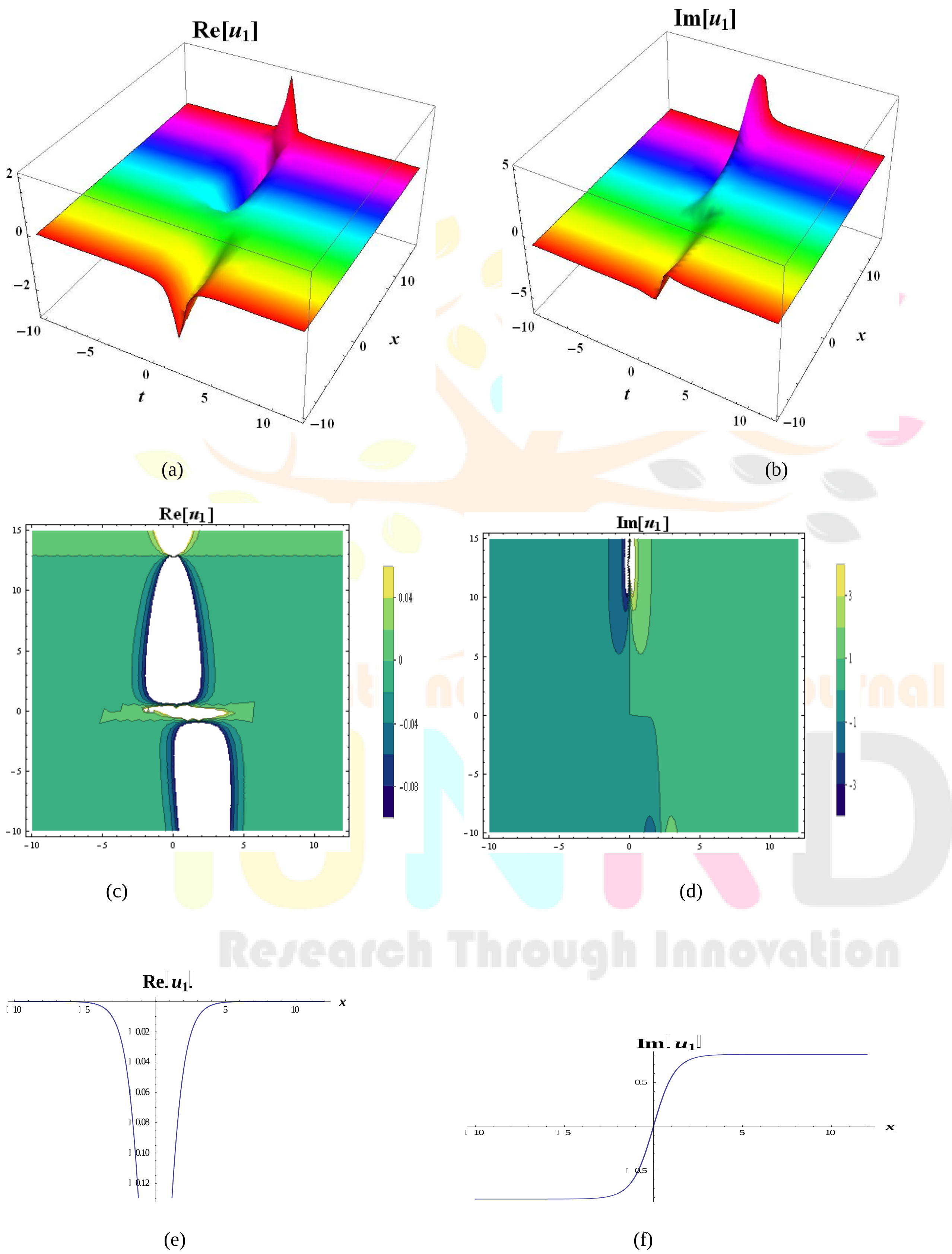

Figure1 The 3D contour and 3D surfaces of Eq. (23) for combined dark- bright soliton solution.

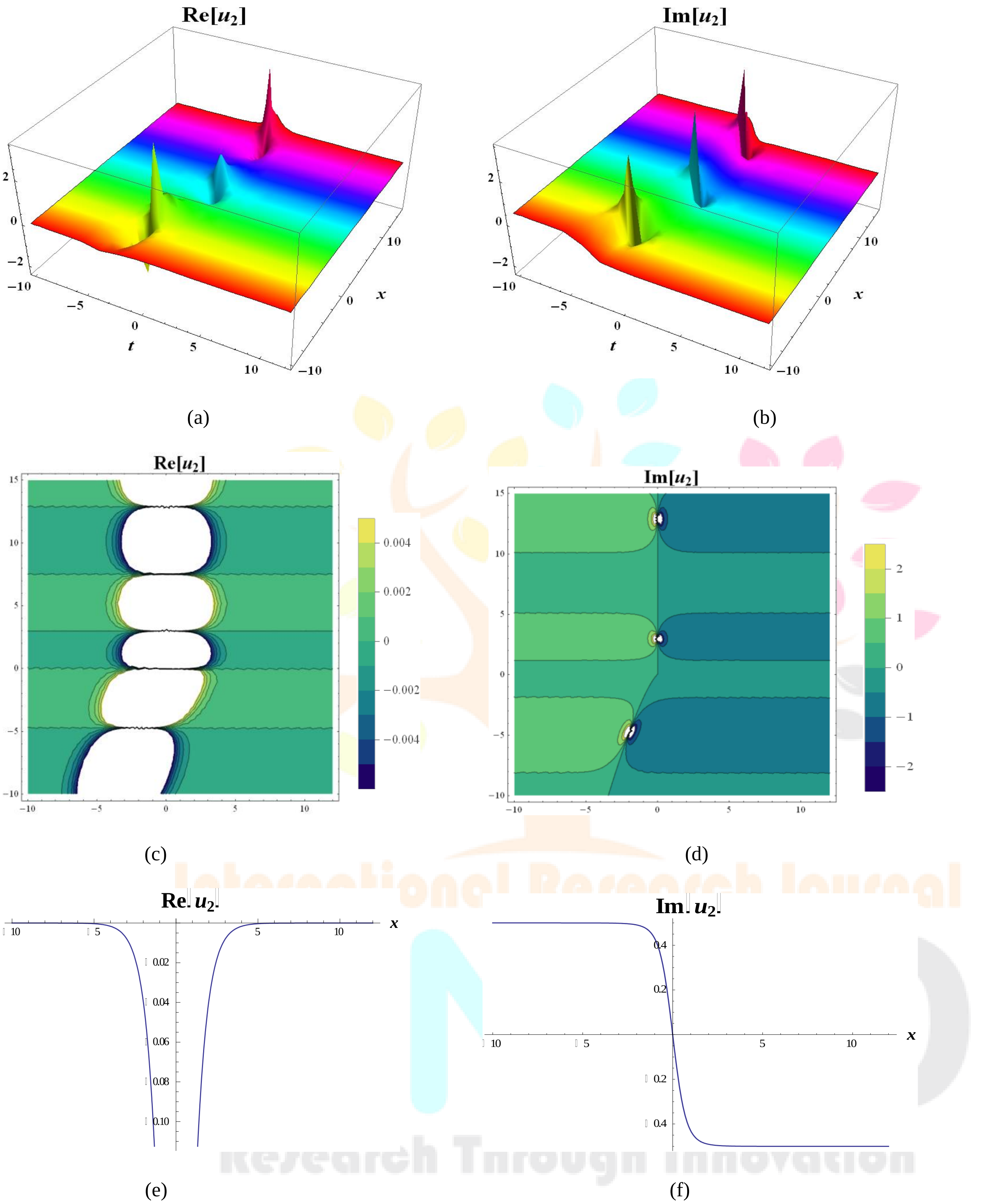

Figure2. The 3D contour and 3D surfaces of Eq. (24) for new dark solution.

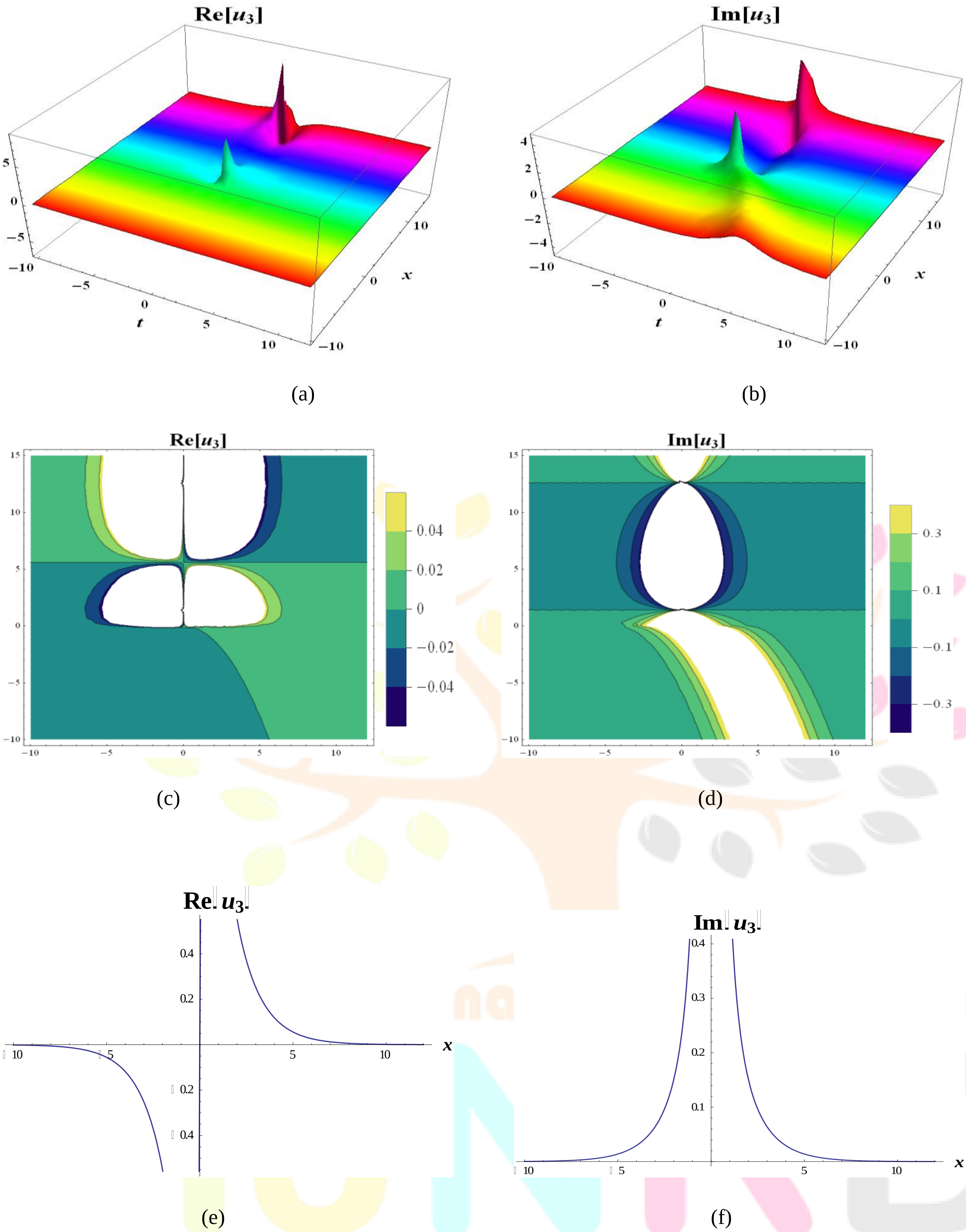

Figure3. The 3D contour and 3D surfaces of Eq. (25) for new singular solution as

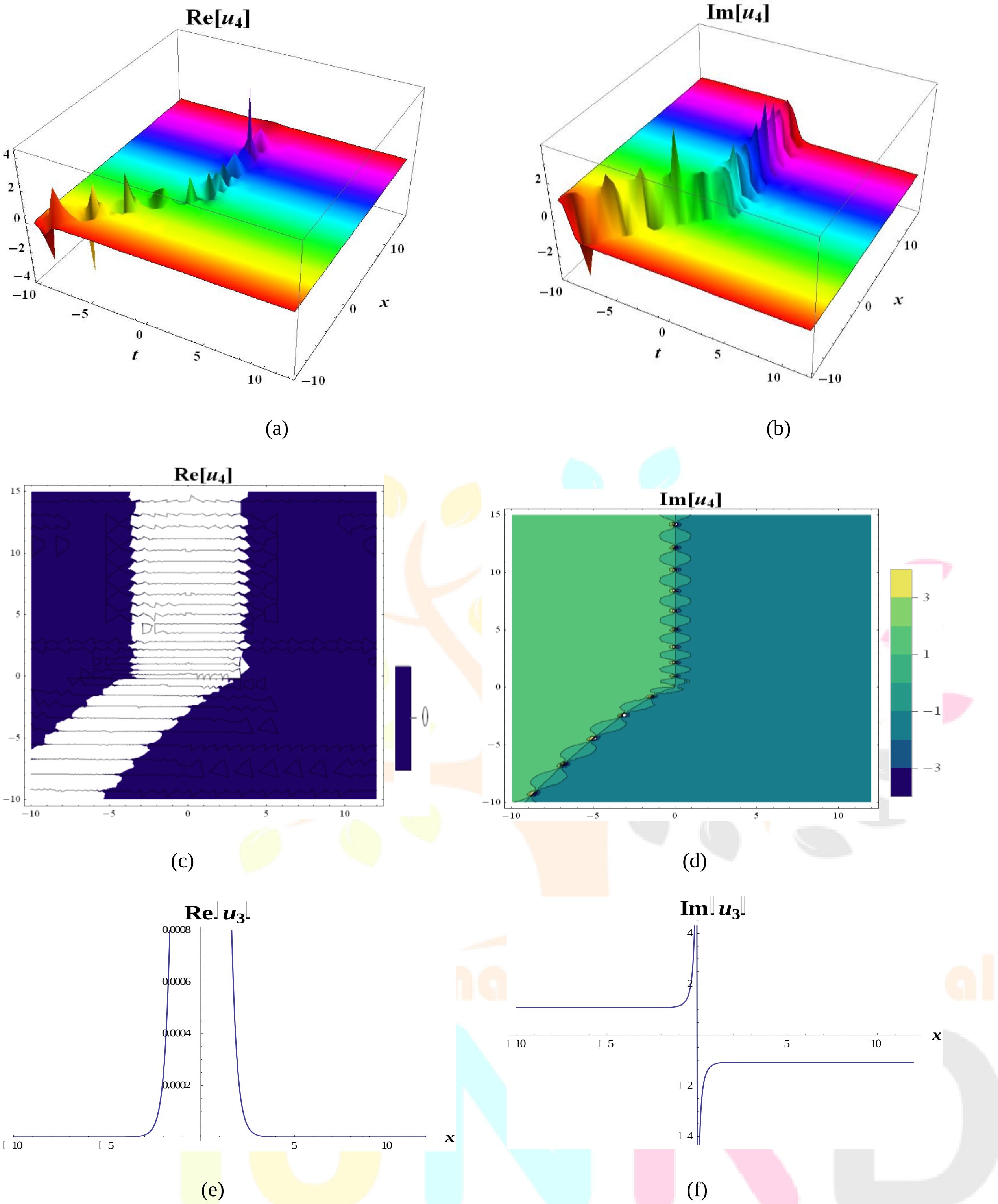

Figure4. The 3D contour and 3D surfaces of Eq. (23) for mixed complex dark-bright solution.

6. Conclusion

This paper studies on the fractional reaction-diffusion equation with conformable which defines to explain wave propagation. By using SGEM, we reach the some new dark, bright, singular solitons and complex wave solutions. All the found wave solutions in this study are entirely new and they have satisfied the fractional reaction-diffusion equation with conformable. Under the suitable chosen of the values of parameters, we plotted 2D, 3D and contour simulations of the wave solutions. From these Figures (1–4), it may be observed that wave solutions to the studied nonlinear model show the estimated wave propagations.

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