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The Structural study of 1-D and 2-D Polytropic Fluids

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ABSTRACT

Theoretical studies for the problem of the equilibrium structure of a rotating gaseous sphere are often carried out to understand the nature of internal structures responsible for various observed phenomena of the stars and it have been a very important investigation in the field of many astrophysicists. The polytropic models have been used for providing the approximate physical representations for a variety of classical astrophysical objects such as stars, planets and globular clusters which is a simplified model. The purpose of the present study is to find the structure of self-gravitating gaseous distribution in the form of 1-D and 2-D shaped Polytropic fluids. The series solution method has been applied to find the general series of Lane- Emden equation and, first time Ramanujan's method has been applied to the obtain the boundary values. Different parameters subjected to the polytropic index *n* have been calculated for n = 0.0, 0.5, 1.0, 1.5, 3.0, 5.0, and 8.0 and the obtained numerical values is compared with the values of Anandaram (2019). The comparison shows that the Ramanujan's method is very efficient and accurate in solving most of the linear differential equation.

Keywords: Cylindrical Polytropic fluid, Ramanujan's method.

INTRODUCTION

The theory of polytropes has been most enthusiastic and importance topic in the field of Mathematics, Astronomy and Astrophysics. The self- gravitating gaseous distribution has been detected over many decades in many parts of the galaxy. The approximate shape of these structure is in the form of thin sheet like masses, long cylindrical rings and spheres. The classical study of the theory of polytropic fluids, investigated by Chandrashekhar (1939) has been extended by J. Ostriker (1964) for cylindrical polytropic fluids. In previous studies Rander (1942) pointed that the infinite cylinders provide to first term of natural series expansion developing the theory of equilibrium of cylindrical rings.

Consider a cylinder of finite radius and infinite length containing the polytropic fluid in a state of self- gravitating hydrostatic equilibrium. The aim of the present problem is to determine the physical variables having the only radial dependence from the axis of the cylinder. The equilibrium theory of self -gravitating infinite cylinders with an underlying equation of state is given by the important relations for the stellar structure which are;

$$\frac{dM(r)}{dP} = 4\pi r^2 \rho \qquad (1)$$

$$\frac{dP}{dr} = -G\frac{M(r)\rho}{r^2} \qquad (2)$$

$$\frac{dL(r)}{dr} = 4\pi r^2 \rho\epsilon \qquad (3)$$

$$P = \frac{k}{m}\rho T \qquad (4)$$

$$P = constant \times \rho^{\gamma} \qquad (5)$$

where; γ, m, ϵ can be calculated as the function of composition and the local condition. The above five equations for the five unknown function $M(r), L(r), T, P$ *and* ρ and mathematical theory that indicates that it should be possible to solve them.

Consider an infinitely long cylinder of finite radius containing polytropic fluid in a state of self-gravitating hydrostatic equilibrium,

$$\nabla V = \frac{1}{\rho}\nabla P \qquad (6)$$

where, ∇V is the gravitational potential within the polytropic cylinder and P and ρ are the pressure and density. From eqn (5); let us suppose that, the distribution of gaseous matter, in a configuration is said to be polytropic fluid if the pressure- density relation at a point is in the form,

$$P = K\rho^{1+\frac{1}{n}} \qquad (7)$$

where k and n are the constants. Such relations can have different possible conditions as,

$n = 0$, represents homogeneous liquid
$n = \frac{3}{2}$, represents a monoatomic gas in adiabatic equilibrium
$n = \infty$, represents an isothermal perfect gas

Substituting eqn (6) in eqn (5) we get,

$$\nabla V = \nabla K(n+1)\rho^{\frac{1}{n}} \qquad (8)$$

On integrating the above eqn (8) gives,

$$V = V_0 + k(n+1)(\rho^{\frac{1}{n}} - \rho_0^{\frac{1}{n}}) \qquad (9)$$

where, subscript zero denotes the value along the axis of cylinder.

If taking divergence of eqn (8) and 2-D Poisson's equation,

$$\frac{1}{r}\frac{d}{dr}\left(\frac{r}{\rho}\frac{dP}{dr}\right) = -4\pi G\rho \qquad (10)$$

here, for obtaining spherical polytropic fluids [1,2]; the radial parameter has to be taken one degree lesser than that in the 3-D Poisson equation.

Now, consider the dimensionality 'D' of the polytropic model for the 2-D and 3-D Poisson equations respectively, equation (10) can be written as,

$$\frac{1}{r^{(D-1)}}\frac{d}{dr}\left(\frac{r^{D-1}}{\rho}\frac{dP}{dr}\right) = -4\pi G\rho \qquad (11)$$

from the above equation, we derive the fundamental differential equation which governs the static equilibrium of a self-gravitating polytropic fluid of any shape.

It is expedient to define new variables,

$$r = \alpha\xi; \quad \alpha^2 = \left[\frac{(n+1)k\rho_c^{\frac{1}{n}-1}}{4\pi G}\right]; \; \rho = \lambda\theta^n; \; V = \lambda^{\frac{1}{n}}k(n+1)\Omega \qquad (12)$$

for complete polytropes it is convenient to take $\lambda \equiv \rho_0$

from eqn (9), the potential ' Ω ' may be written as;

$$\Omega = \Omega_0 + \theta - 1 \qquad (13)$$

In the present application to an infinitely long cylinder, eqn (13) becomes *(Ref Anandaram[5])*,

$$\frac{1}{\xi^{D-1}}\frac{d}{d\xi}\left(\xi^{D-1}\frac{d\theta}{d\xi}\right) = \theta^n + (D-1)\frac{1}{\xi}\theta' = -\theta^n \qquad (14)$$

Here, $\theta \equiv \theta(\xi) \equiv \theta_n(\xi)$ is the Emden function for the cylindrical polytropic fluid of index 'n'.

SPECIAL CASE:

- When D =3 the equation represents the structure of 2-D polytropic fluid,

$$\frac{1}{\xi^2}\frac{d}{d\xi}\left(\xi^2\frac{d\theta}{d\xi}\right) = \theta'' + \frac{2}{\xi}\theta' = -\theta^n$$

- When D = 2, the equation represents the structure of 1-D polytropic fluid,

$$\frac{1}{\xi}\frac{d}{d\xi}\left(\xi\frac{d\theta}{d\xi}\right) = \theta'' + \frac{1}{\xi}\theta' = -\theta^n$$

BOUNDARY CONDITION:

- It can be specified that, at origin from the requirement that $\lambda \equiv \rho_0$, θ is finite at $\xi = 0$, the conditions are,

$$\theta(0) = 1 \text{ and } \theta'(0) = 0 \qquad (15)$$

- The surface of the cylinder is defined by the condition $P = \rho = 0$; the radius of the cylinder ξ_1 is defined by,

$$\theta(\xi_1) = 0 \qquad (16)$$

Therefore, the series solution of the generalised Lane-Emden equation of polytropic configuration for cylinders in gravitational equilibrium is given by,

- For 2-D polytropic configuration,

$$\psi_0 = 1 - \frac{1}{4}\xi^2 + \frac{n}{64}\xi^4 - \frac{n(3n-2)}{2304}\xi^6 + \frac{n(12-29n+18n^2)}{147456}\xi^8 \mp \ldots \qquad (17)$$

- For 1-D polytropic configuration,

$$\psi_0 = 1 - \frac{1}{2}\xi^2 + \frac{n}{24}\xi^4 - \frac{n(4n-3)}{720}\xi^6 + \frac{n(30-63n+34n^2)}{40320}\xi^8 \mp \ldots \qquad (18)$$

THE SERIES SOLUTION FOR CYLINDRICAL LANE EMDEN EQUATION

In the special case, when the values of $n = 0$ *and* 1, eqn (17) is linear and the required solution can be written as,

Case I: The solution is connected with the potential liquid cylinder, when

$$n = 0; \quad \theta = 1 - \frac{1}{4}\xi^2, \text{ where } \xi_1 = 2.000000 \qquad (19)$$

$$n = 1; \theta = 1 - \frac{1}{4}\xi^2 + \frac{n}{64}\xi^4 - \frac{n(3n-2)}{2304}\xi^6 + \frac{n(12-29n+18n^2)}{147456}\xi^8 \mp \ldots,$$

where $\xi_1 = 2.404333$ (20)

Case II: The height of either surface from the central plane of the polytropic sheet like masses can be,

$$n = 0;\ \ \theta = 1 - \frac{1}{2}\xi^2, \text{ where } \xi_1 = 1.414213 \qquad (21)$$

$$n = 0;\ \ \theta = 1 - \frac{1}{2}\xi^2 + \frac{n}{24}\xi^4 - \frac{n(4n-3)}{720}\xi^6 + \frac{n(30-63n+34n^2)}{40320}\xi^8 \mp \ldots$$

where $\xi_1 = 1.570795$ (22)

for all the other values of n, no analytic solution is available. The series solution of the generalised 1-D Lane-Emden equation of cylindrical polytropic configuration for each value of n is given by,

$$\theta(\xi) = 1 - \frac{1}{2}\xi^2 + \frac{n}{24}\xi^4 - \frac{n(4n-3)}{720}\xi^6 + \frac{n(30-63n+34n^2)}{40320}\xi^8 \mp \ldots$$

solving the above eqn by applying Ramanujan's method;

$$[f(\xi) = 1 - \left(\frac{1}{2}\xi^2 + \frac{n}{24}\xi^4 - \frac{n(4n-3)}{720}\xi^6 + \frac{n(30-63n+34n^2)}{40320}\xi^8 \mp \cdots.\right)]^{-1} =$$

$$b_1\xi^2 + b_2\xi^4 + b_3\xi^6 + b_4\xi^8 + \ldots\ldots$$

$$a_1 = \frac{1}{2},\ a_2 = -\frac{1}{24},\ a_3 = \frac{1}{720}, a_4 = -\frac{1}{40320}$$

$$b_1 = 1,\ b_2 = a_1 = \frac{1}{2} = 0.5,\ b_3 = a_2 + a_1^2 = a_2 b_1 + a_1 b_2 = 0.2083333 \ldots b_{10} = 0.000376$$

The ratio of b_n/b_{n-1}, of the above equation determines the smallest root (ξ) of the equation.

$$\frac{b_1}{b_2} = 1.414214,\quad \frac{b_2}{b_3} = 1.549193, \ldots\ldots\ldots\ \frac{b_6}{b_{17}} = 1.570795$$

Hence, 1.570795 is the root of equation (22) for the polytropic index $n = 1$.

RESULT & DISCUSSION

The series solution for 1-D and 2-D Cylindrical polytropic fluid equation

$$\frac{1}{\xi}\frac{d}{d\xi}\left(\xi\frac{d\theta}{d\xi}\right) = \theta'' + \frac{1}{\xi}\theta' = -\theta^n$$

is calculated for different polytropic index i.e $n = 0.0, 0.5, 1.0, 1.5, 2.0$ *and* 3.0 by using computational method. Further, the boundary values ξ = 0 at $\theta(0) = 1$ and $'(0) = 0$ for small values of ξ has been obtained using Ramanujan's method and it is compared with the Anandaram's numerical result, as presented in the table. The % error shows the efficiency of solving such equations by Ramanujan's method.

Table I: Comparison table for the boundary values (ξ_1) using Ramanujan's method with Anandaram's values for 2-D polytropic fluid

n	Anandaram's Result	Author's result	% error
0.0	2.000000000	2.000000000	0.000000
0.5	2.189662199	2.184779513	0.222988
1.0	2.404825557	2.404333986	0.020441
1.5	2.647776766	2.656844657	0.342472
2.0	2.921320724	2.921186973	0.004578
3.0	3.573900983	3.577708764	0.106544

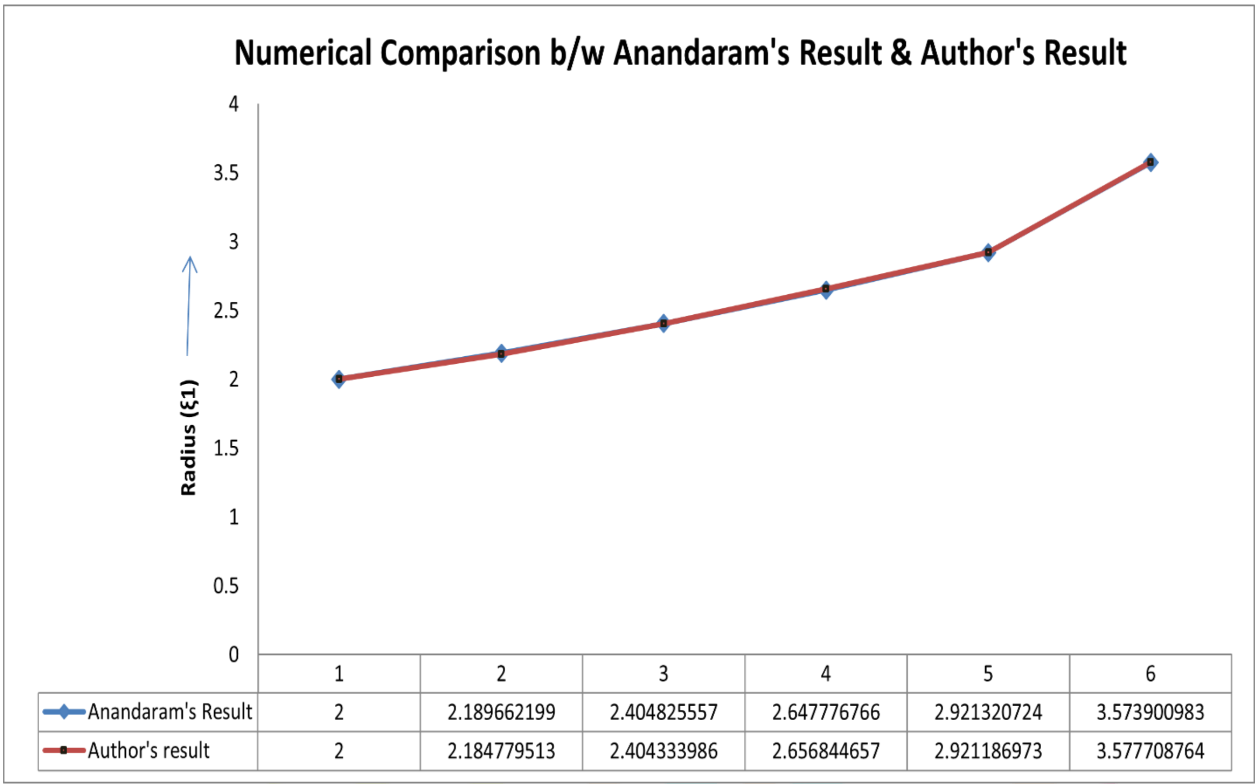
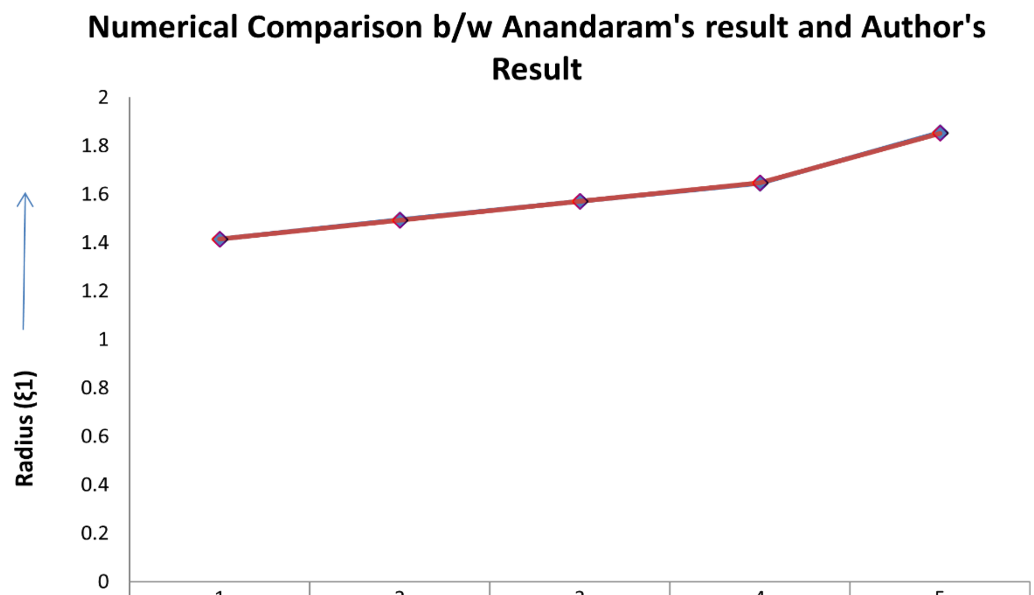


Table II: Comparison table for the boundary values (ξ_1) using Ramanujan's method with Anandaram's values for 1-D polytropic fluid.

n	Anandaram's value (ξ_1)	Author's value ((ξ_1)	% Error
0.0	1.41421356	1.41421356	0.000000
0.5	1.4936684	1.49199	0.112494
1.0	1.57079633	1.570795277	6.7E-05
1.5	1.64534085	1.647317938	0.120019
3.0	1.85407468	1.8516402	0.131477



	1	2	3	4	5
ξ1 (Anandaram's value)	1.41421356	1.4936684	1.57079633	1.64534085	1.85407468
Author's value (ξ1)	1.41421356	1.49199	1.570795277	1.647317938	1.8516402

SUMMARY & CONCLUSION

In this paper, we have found the technique of the computational method provides a good way for obtaining analytical solutions of linear differential equations, with prescribed boundary conditions. The Ramanujan's method looks expedient and safe from view point of astrophysical application. The main advantages of Ramanujan's methods are:

- The solution is far more exact than the series solution and are very close to the numerical solutions.
- The boundary values of physical parameters are easily determinable even without use of complicated computer software.
- A simple calculator and Microsoft excel sheet can be used to obtain the series solution by using Ramanujan's method.

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