IJNRD.ORG ISSN : 2456-4184
INTERNATIONAL JOURNAL OF NOVEL RESEARCH AND DEVELOPMENT (IJNRD) | IJNRD.ORG
An International Open Access, Peer-reviewed, Refereed Journal

PYTHAGOREAN FUZZY TOPOLOGICAL STRUCTURES CONNECTED WITH COMPACT HAUSSDORFF SPACE

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***Abstract*:** In this paper, we study the concept of Pythagorean fuzzy topological properties of such algebras such as connectedness, strong connectedness and compact Haussdorff space. We also obtain the characteristic of the homomorphic image and inverse image of Pythagorean fuzzy topological BCC-ideals (BCK-ideals) of BCC-algebras (BCK-algebras).

***2020 AMS classification*:** 03G25, 06F35, 08A72.

Keywords: binary operations, BCK-algebra, BCC-algebra, Pythagorean fuzzy set, Pythagorean fuzzy topology, homomorphism, pre image, image, BCC-ideal, connected.

1.Introduction: In classical set theory, the membership of elements in a set is assessed in binary terms according to a bivalent condition: an element either belongs or does not belong to the set. As an extension, fuzzy set theory (See [22]) permits the gradual assessment of the membership of elements in a set; this is described with the aid of a membership function valued in the real unit interval [0; 1]. As a generalization of fuzzy set, Atanassov [1] created intuitionistic fuzzy set. Intuitionistic fuzzy set is widely used in all _elds (See [4, 5, 12, 18] for applications in algebraic structures). In 2013, Yager [19, 20, 21] introduced Pythagorean fuzzy set and compared it with intuitionistic fuzzy set. Pythagorean fuzzy set is a new extension of intuitionistic fuzzy set that conducts to simulate the vagueness originated by the real case that might arise in the sum of membership and non-membership is bigger than 1. Pythagorean fuzzy set is applied to groups (See [2]), UP-algebras (See [15]) and topological spaces (See [14]). Senapati et al. [16] introduced Fermatean fuzzy set which is another extension of intuitionistic fuzzy sets and it is applied to groups (See [17]). Ibrahim et al. [9] introduced fermatean fuzzy sets and applied it to topological spaces. In this paper, we study the concept of Pythagorean fuzzy topological properties of such algebras such as connectedness, strong connectedness and compact Haussdorff space. We also obtain the characteristic of the homomorphic image and inverse image of Pythagorean fuzzy topological BCC-ideals (BCK-ideals) of BCC-algebras (BCK-algebras).

2. Preliminaries of BCC-algebras(BCK-algebras)

In this section, we first review some definitions and properties which will be used in the sequel.

A non-empty set G with a constant 0 and binary operation $*$ is called a BCC-algebra if it satisfies the following conditions:

a) $\left(\left((x*y)*(z*y)\right)*(x*y)=0\right)$

b) $x*x=0$

c) $0*x=0$

d) $x*0=0$

e) $x*y=0\,,y*x=0\;\Rightarrow x=y$

for all $x,y,z\in G$. In BCC-algebra, the following equality holds $(x*y)*x=0$.

Obviously, any BCK-algebra is BCC-algebra but there exist BCC-algebras which are not necessarily BCK-algebra. We note that a BCC-algebra is BCK-algebra if and if only, it satisfies the equality $(x*y)*z=(x*z)*y$.

A non-empty subset 'S' of a BCK-algebra 'G' is called a sub algebra of G if it is closed under the BCC-operation. Such algebra contains the constant 0 and it is clearly a BCC-algebra, but some sub algebras may be also BCK-algebras. Moreover, there exit BCC-algebras which all sub algebras are BCK-algebras.

A mapping $\varphi: G_1\rightarrow G_2$ of BCC-algebras is called a homomorphism if $\varphi(x*y)=\varphi(x)*\varphi(y)$ holds, for all $x,y\in G_1$.

For a non-empty given set G, let I be the closed unit interval [0, 1]. Then, an fermatean fuzzy set is an object of the form $A=\left\{\langle x,{\delta_A}^2(x),{\lambda_A}^2(x)\rangle/\,x\in G\right\}$, when the mappings ${\delta_A}^2: G\rightarrow I$ and ${\lambda_A}^2: G\rightarrow I$ denote the degree of membership (namely, $\delta_A(x)$) and the degree of non-membership (namely, $\lambda_A(x)$) of each element $x\in G$ to the object 'A' respectively satisfying $0\le{\delta_A}^2(x)+{\lambda_A}^2(x)\le 1$ for all $x\in G$.

The complement of the fermatean set is $A^C=\left\{\langle x,{\lambda_A}^2(x),{\delta_A}^2(x)\rangle/\,x\in G\right\}$. Obviously, every fuzzy A on a non-empty G is an Pythagorean fuzzy set of the form $A=\left\{\langle x,{\delta_A}^2(x),1-{\lambda_A}^2(x)\rangle/\,x\in G\right\}$. For the sake of simplicity, we just write $A=\langle{\delta_A}^2,{\lambda_A}^2\rangle$ instead of $A=\{\langle x,\delta_A(x),\lambda_A(x)\rangle/\,x\in G\}$.

The fermatean fuzzy sets $0\sim$ and $1\sim$ in G are defined by $0\sim=\{\langle x,0,1\rangle: x\in G\}$ and $1\sim=\{\langle x,1,0\rangle: x\in G\}$, respectively.

If φ is a mapping which maps a set G_1 into another set G_2, then the following statement hold:

(a) If $B=\left\{\langle y,{\delta_B}^2(y),{\lambda_B}^2(y)\rangle/\,y\in G_2\right\}$ is an Pythagorean fuzzy set in G_2, then the pre image of B under φ, denoted by $\varphi^{-1}(B)$, is still an Pythagorean fuzzy set in G_1, we now write $\varphi^{-1}(B)=\{\langle x,\ \varphi^{-1}(\delta_B)(x),\ \varphi^{-1}(\lambda_B)(x)\rangle/\,x\in G_1\}$.

(b) If $A=\left\{\langle x,{\delta_A}^2(x),{\lambda_A}^2(x)\rangle/\,x\in G_1\right\}$ in an Pythagorean fuzzy set in G_1, then the image of A under φ, denoted by $\varphi(A)$, is also an Pythagorean fuzzy set in G_2, which is defined by $\varphi(A)=\left\{\langle y,\varphi_{sup}(\delta_A)(y),\varphi_{inf}(\lambda_A)(y)\ \rangle: y\in G_2\right\}$, where

$$\varphi_{sup}(\delta_A)(y) = \begin{cases} \sup_{x \in \varphi^{-1}(y)} \delta_A(x), & \text{if } \varphi^{-1}(y) \neq 0, \\ 0, & \text{else where,} \end{cases}$$

$$\varphi_{inf}(\lambda_A)(y) = \begin{cases} \inf_{x \in \varphi^{-1}(y)} \lambda_A(x), & \text{if } \varphi^{-1}(y) \neq 0, \\ 0, & \text{else where,} \end{cases}$$

for each $y \in G_2$.

Proposition-2.1: Let A, $A_i (i \in I)$be Pythagorean fuzzy set in G_1 and B an Pythagorean fuzzy set in G_2. If $\varphi: G_1 \to G_2$ is a function, then the following properties hold for the function φ:

(a) If φ is surjective, then $\varphi(\varphi^{-1}(B)) = B$.

(b) $\varphi^{-1}(\cup_{i=1}^{n} A_i) = \cup_{i=1}^{n} \varphi^{-1}(A_i)$.

(c) $\varphi^{-1}(1\sim) = 1\sim$.

(d) $\varphi^{-1}(0\sim) = 0\sim$.

(e) $\varphi(1\sim) = 1\sim$, if φ is surjective

(f) $\varphi(0\sim) = 0\sim$.

Definition-2.2: An Pythagorean fuzzy topology on a non-empty set G is a family τ of Pythagorean fuzzy sets in G which satisfies the following conditions:

(i) $0\sim, 1\sim \in \tau$.

(ii) If $G_1, G_2 \in \tau$, then $G_1 \cap G_2$.

(iii) If $G_j \in \tau$ for all $j \in J$, then $\cup_{i \in I} G_i \in \tau$.

The pair (G, τ) is called an Pythagorean fuzzy topological space and any Pythagorean fuzzy set in τ is called an fermatean fuzzy open sets in G. The topology τ on a Pythagorean fuzzy topological space is said to be an indiscrete fermatean fuzzy topology if it's only element are $0\sim$ and $1\sim$. On the other hand, Pythagorean fuzzy topology τ on a space G is said to be discrete Pythagorean fuzzy topology
if the topology fermatean fuzzy topology τ contains all fermatean fuzzy subsets of G.

If A is an Pythagorean fuzzy set in an Pythagorean fuzzy topological space (G, τ), then the induced Pythagorean fuzzy topological space on A is the family of Pythagorean fuzzy sets in A which are the intersection with A of Pythagorean fuzzy sets in G. The induced Pythagorean fuzzy topology is denoted by τ_A, and the pair (A, τ_A) is called an fuzzy subspace of (G, τ).

Let (G_1, τ_1) and (G_2, τ_2) be two Pythagorean fuzzy topological spaces and
$\varphi: (G_1, \tau_1) \to (G_2, \tau_2)$ a function. Then φ is said to be Pythagorean fuzzy continuous function if and only if the pre image of each Pythagorean fuzzy set in τ_2 is an Pythagorean fuzzy set in τ_1. Let(G_1, τ_1) and (G_2, τ_2) be two Pythagorean fuzzy topological spaces and $\varphi: (G_1, \tau_1) \to (G_2, \tau_2)$ a function. Then φ is said to be Pythagorean n fuzzy open if and only if the image of each Pythagorean fuzzy set in τ_1 is an Pythagorean fuzzy set in τ_2.

3. Pythagorean fuzzy topological sub algebras

***Definition*-3.1:** A Pythagorean fuzzy set $A = \langle {\delta_A}^2, {\lambda_A}^2 \rangle$ in G is called Pythagorean fuzzy sub algebra of G if it satisfies the following conditions;

PFS1 : ${\delta_A}^2(x * y) \geq \min\{{\delta_A}^2(x), {\delta_A}^2(y)\}$

PFS2 : ${\lambda_A}^2(x * y) \leq \max\{{\lambda_A}^2(x), {\lambda_A}^2(y)\}$, for all $x, y \in G$.

Example-3.2: Let $G = \{0, 1, 2, 3, 4\}$ be a BCC-algebra with the following Cayley table.

+	0	1	2	3	4
0	0	0	0	0	0
1	1	0	1	0	0
2	2	2	0	0	0
3	3	3	1	0	0
4	4	3	4	3	0

Let $A = \langle {\delta_A}^2, {\lambda_A}^2 \rangle$ be an Pythagorean fuzzy set in G defined by ${\delta_A}^2(4) = 0.07$, ${\delta_A}^2(x) = 0.6$, ${\lambda_A}^2(x) = 0.5$ and ${\lambda_A}^2(4) = 0.06$ for all $x \neq d$. Then A is Pythagorean fuzzy sub algebra of G.

Definition-3.3: Let τ_1 and τ_2 be an Pythagorean fuzzy topologies on BCC-algebras G_1 and G_2 respectively. A function $\varphi: (G_1, \tau_1) \to (G_2, \tau_2)$ is called an Pythagorean fuzzy continuous function which maps (G_1, τ_1) and (G_2, τ_2) if φ satisfies the following conditions:

(i) For every $A \in \tau_2$, $\varphi^{-1}(A) \in \tau_1$.

(ii) For every Pythagorean fuzzy sub algebra A (of G_2) in τ_2, $\varphi^{-1}(A)$ is Pythagorean fuzzy sub algebra (of G_1) in τ_1.

Proposition-3.4: If in τ_1 is an Pythagorean fuzzy topology on a BCC-algebra G_1 and τ_2 is an Pythagorean fuzzy topology on a BCC-algebra G_2, then every function $\varphi: (G_1, \tau_1) \to (G_2, \tau_2)$ is a Pythagorean fuzzy continuous function.

Proof: Since τ_2 is an indiscrete fPythagorean fuzzy topology, $\tau_2 = (0\sim, 1\sim)$..

Let $\varphi: G_1 \to G_2$ be any mapping of BCC-algebras. Then, every member of τ_2 is an Pythagorean fuzzy topology on a BCC-algebra G_2.

We now show that φ is Pythagorean fuzzy continuous function. We only need to prove that for every $A \in \tau_2$, $\varphi^{-1}(A) \in \tau_1$.

For this purpose, we let $0\sim \in \tau_2$. Then for any $x \in G_1$, we have

$\varphi^{-1}(0\sim)(x) = 0\sim(\varphi(x)) = 0 = 0\sim(x)$. This show that $(\varphi^{-1}(0\sim)) = 0\sim \in \tau_1$.

On the other hand, if $1\sim \in \tau_2$ and $x \in G_1$, then

$\varphi^{-1}(1\sim)(x) = 1\sim(\varphi(x)) = 1 = 1\sim(x)$. Thus $(\varphi^{-1}(1\sim)) = 1\sim \in \tau_1$.

This show that φ is indeed an Pythagorean fuzzy continuous function of G_1 to G_2.

Theorem-3.5: Let τ_1 and $\tau_2$be any two discrete Pythagorean fuzzy topologies defined on the BCC-algebras G_1 and G_2 respectively. Then every homomorphism $\varphi: (G_1, \tau_1) \to (G_2, \tau_2)$ is an Pythagorean fuzzy continuous function.

Proof: Since τ_1 and τ_2 are discrete Pythagorean fuzzy topologies on the BCC-algebras G_1 and G_2 respectively, we have $\varphi^{-1}(A) \in \tau_1$ for every $A \in \tau_2$.

We note that φ is not the usual inverse homomorphism from G_2 to G_1.

Let $A = \langle {\delta_A}^2, {\lambda_A}^2 \rangle$ be an Pythagorean fuzzy sub algebra (of G_2) in τ_2. Then for $x, y \in G_1$, we have, $\left(\varphi^{-1}({\delta_A}^2)\right)(x * y) = {\delta_A}^2(\varphi(x * y))$

$$= {\delta_A}^2(\varphi(x) * \varphi(y))$$

$$\geq \min\{{\delta_A}^2(\varphi(x)), {\delta_A}^2(\varphi(y))\}$$

$$= \min\left\{\left(\varphi^{-1}({\delta_A}^2)\right)(x), \left(\varphi^{-1}({\delta_A}^2)\right)(y)\right\} \text{ and}$$

$$\left(\varphi^{-1}({\lambda_A}^2)\right)(x * y) = {\lambda_A}^2(\varphi(x * y))$$

$$= {\lambda_A}^2(\varphi(x) * \varphi(y))$$

$$\leq \max\{{\lambda_A}^2(\varphi(x)), {\lambda_A}^2(\varphi(y))\}$$

$$= \max\left\{\left(\varphi^{-1}({\lambda_A}^2)\right)(x), \left(\varphi^{-1}({\lambda_A}^2)\right)(y)\right\}$$

Hence $\varphi^{-1}(A)$ is an Pythagorean fuzzy sub algebra (of G_1) in τ_1 and consequently, φ is an Pythagorean fuzzy continuous function which maps (G_1, τ_1) to (G_2, τ_2).

Definition-3.6: Let (G_1, τ_1) and (G_2, τ_2) be Pythagorean fuzzy topology sub algebras. A function $\varphi: (G_1, \tau_1) \to (G_2, \tau_2)$ is said to be an Pythagorean fuzzy homomorphism if it satisfies the following conditions:

- φ is an injective and surjective function.
- φ is fuzzy continues function which maps G_1 to G_2.
- φ^{-1} is fuzzy continues function which maps G_2 to G_1.

Definition-3.7: Let τ be an Pythagorean fuzzy topology of BCC-algebra G. An Pythagorean fuzzy topology (G, τ) is an Pythagorean fuzzy Hausdorff space if and only if for any discrete Pythagorean fuzzy point $x_1, x_2 \in G$, there exits Pythagorean fuzzy topology $F_1 = \langle \delta^3_{F_1}, \lambda^2_{F_1} \rangle$ and $F_2 = \langle \delta^2_{F_2}, \lambda^2_{F_2} \rangle$ such that $\delta^2_{F_1}(x_1) = 1$, $\lambda^2_{F_1}(x_1) = 0$, $\delta^2_{F_2}(x_2) = 1$, $\lambda^2_{F_2}(x_2) = 0$ and $F_1 \cap F_2 = 0\sim$.

Theorem-3.8: Let τ_1 and τ_2 be Pythagorean fuzzy topologies on the BCC-algebras G_1 and G_2 respectively and let $\varphi: (G_1, \tau_1) \to (G_2, \tau_2)$ be an Pythagorean fuzzy homomorphism. Then G_1 is an Pythagorean fuzzy Hausdorff space if and only if G_2 is an Pythagorean fuzzy Hausdorff space.

Proof: Suppose that G_1 is a Pythagorean fuzzy Hausdorff space.

Let x_1, x_2 be the Pythagorean fuzzy point in τ_2 with $x \neq y$ where $x, y \in G_1$.

Then $\varphi^{-1}(x) \neq \varphi^{-1}(y)$ because φ is injective function.

For $z \in G_1$, $\left(\varphi^{-1}(x_1)\right)(z) = x_1\left(\varphi(z)\right)$

$$= \begin{cases} s \in [0,1], & \text{if } \varphi(z) = x \\ 0, & \text{if } \varphi(z) \neq x \end{cases} = \begin{cases} s \in [0,1], & \text{if } z = \varphi^{-1}(x) \\ 0, & \text{if } z \neq \varphi^{-1}(x) \end{cases}$$

$$= \left(\varphi^{-1}(x)\right)_1(z).$$

That is, $\left(\varphi^{-1}(x_1)\right)(z) = \left(\varphi^{-1}(x)\right)_1(z)$ for all $z \in G$. Hence $\varphi^{-1}(x_1) = \left(\varphi^{-1}(x)\right)_1$.

Similarly we can also prove that $\varphi^{-1}(x_2)=\left(\varphi^{-1}(x)\right)_2$. Now by the definition of an Pythagorean fuzzy Hausdorff space, there exist Pythagorean fuzzy order F_1 and F_2 of $\varphi^{-1}(x_1)$ and $\varphi^{-1}(x_2)$ respectively such that $F_1 \cap F_2 = 0\sim$. Since φ is an Pythagorean fuzzy continuous map from G_2 to G_1, there exist Pythagorean fuzzy orders $\varphi(F_1)$ and $\varphi(F_2)$ of x_1 and x_2 respectively such that $\varphi(F_1) \cap \varphi(F_2) = \varphi(F_1 \cap F_2) = \varphi(0\sim) = 0\sim$. This implies that G_2 is a Pythagorean fuzzy Hausdorff space.

Conversely, if (G_2, τ_2) is a Pythagorean fuzzy Hausdorff space, then by using a similar argument as above and by the fact that both φ and φ^{-1} are Pythagorean fuzzy continuous functions, we can easily prove that (G_1, τ_1) is an Pythagorean fuzzy Hausdorff space. Hence the proof.

Definition-3.9: Let τ be an Pythagorean fuzzy topology on a BCC-algebra G. Then (G, τ) is called an Pythagorean fuzzy C_5-disconnected space if there exists an Pythagorean fuzzy open and closed set F such that $F \neq 0\sim$ and $F \neq 1\sim$.

Theorem-3.10: Let τ_1 and τ_2 be the Pythagorean fuzzy topology sub algebras G_1 and G_2 respectively and let $\varphi: G_1 \to G_2$ be an Pythagorean fuzzy continuous surjective function. If (G_1, τ_1) is an Pythagorean fuzzy C_5-connected space then (G_2, τ_2) is also an Pythagorean fuzzy C_5-connected space.

Proof: Assume that (G_2, τ_2) is a Pythagorean fuzzy C_5-disconnected. Then there exist an Pythagorean fuzzy open and closed set F such that $F \neq 0\sim$ and $F \neq 1\sim$.

Since φ is an (3, 2)- fuzzy continuous function $\varphi^{-1}(F)$ is both Pythagorean fuzzy open and Pythagorean fuzzy closed set. In this case $\varphi^{-1}(F) \neq 0\sim$ or $\varphi^{-1}(F) \neq 1\sim$.

Since, $F = \varphi\left(\varphi^{-1}(F)\right) = \varphi(0\sim) = 0\sim$ and $F = \varphi\left(\varphi^{-1}(F)\right) = \varphi(1\sim) = 1\sim$.

We see that these results contradict to our assumption.

Hence the space (G_2, τ_2) must be Pythagorean fuzzy C_5-connected space.

Definition-3.11: Let τ be an Pythagorean fuzzy topology on a BCC-algebra G. An Pythagorean fuzzy topology (G, τ) is called an Pythagorean fuzzy disconnected space if there exist fermatean fuzzy open sets $A \neq 0\sim$ and $B \neq 0\sim$ such that $A \cup B = 0\sim$. Naturally, we call the set (G, τ) Pythagorean fuzzy connected if (G, τ) is not Pythagorean fuzzy disconnected.

Theorem-3.12: Let τ_1 and τ_2 be Pythagorean fuzzy topology set on BCC-algebras G_1 and G_2 respectively and let $\varphi: (G_1, \tau_1) \to (G_2, \tau_2)$ be an Pythagorean fuzzy continuous and surjective function. If G_1 is an Pythagorean fuzzy connected space, then so is G_2.

Proof: Suppose that G_2 is an Pythagorean fuzzy disconnected, then there exists Pythagorean fuzzy open sets $C \neq 0\sim$ and $D \neq 0\sim$ in G_2 such that $C \cup D = 1\sim$ and $C \cap D = 0\sim$.

Since φ is Pythagorean fuzzy continuous function, $A = \varphi^{-1}(C)$ and $B = \varphi^{-1}(D)$ are Pythagorean fuzzy open sets in G_1.

Clearly, $C \neq 0\sim$ implies that $A = \varphi^{-1}(C) \neq 0\sim$, and $D \neq 0\sim$ implies that $B = \varphi^{-1}(D) \neq 0\sim$.

Now $C \cup D = 1\sim$.

$\Rightarrow \varphi^{-1}(C \cup D) = \varphi^{-1}(1\sim)$.

$\Rightarrow \varphi^{-1}(C) \cup \varphi^{-1}(D) = 1\sim$ implies $A \cup B = 1\sim$ and

$C \cap D = 0\sim \Rightarrow \varphi^{-1}(C \cap D) = \varphi^{-1}(0\sim)$

$\Rightarrow \varphi^{-1}(C) \cap \varphi^{-1}(D) = 0\sim$ implies $A \cap B = 0\sim$.

This clearly contradicts our hypothesis.

Hence G_2 is an Pythagorean fuzzy connected space.

Definition-3.13: An Pythagorean fuzzy topology space (G, τ) is said to be an Pythagorean fuzzy strongly connected, if there exists no non-zero Pythagorean fuzzy closed sets A and B in G such that $\delta_A^2 + \delta_B^2 \leq 1$ and $\lambda_A^2 + \lambda_B^2 \geq 1$.

The following fact follows immediately from the definition.

Propositon-3.14: G is Pythagorean fuzzy strongly connected if and only if there exist an Pythagorean fuzzy open sets A and B in G such that $A \neq 1\sim \neq B$ and $\delta_A^2 + \delta_B^2 \geq 1$, $\lambda_A^2 + \lambda_B^2 \leq 1$.

We now formulate the following theorem.

Theorem-3.15: Let τ_1 and τ_2 be Pythagorean fuzzy topology set on BCC-algebras G_1 and G_2 respectively and let $\varphi: (G_1, \tau_1) \to (G_2, \tau_2)$ be an Pythagorean fuzzy continuous and surjective mapping. If G_1 is an Pythagorean fuzzy strongly connected, then so is G_2.

Proof: Suppose that G_2 is not an Pythagorean fuzzy strongly connected. Then there exists Pythagorean fuzzy open sets $C \neq 0\sim$ and $D \neq 0\sim$ so that $\delta_C^2 + \delta_D^2 \leq 1$ and $\lambda_C^2 + \lambda_D^2 \geq 1$. Since φ is an Pythagorean fuzzy continuous function, $\varphi^{-1}(C)$ and $\varphi^{-1}(D)$ are (m, n)-fuzzy closed sets in G_1. Now we can deduce the following equalities;

$$\delta^2_{\varphi^{-1}(C)} + \delta^2_{\varphi^{-1}(D)} = \varphi^{-1}(\delta_C^2) + \varphi^{-1}(\delta_D^2)$$

$$= \delta_C^2 \text{ o } \varphi + \delta_D^2 \text{ o } \varphi \leq 1 \text{ (Since } \delta_C^2 + \delta_D^2 \leq 1),$$

$$\lambda^2_{\varphi^{-1}(C)} + \lambda^2_{\varphi^{-1}(D)} = \varphi^{-1}(\lambda_C^2) + \varphi^{-1}(\lambda_D^2)$$

$$= \lambda_C^2 \text{ o } \varphi + \lambda_D^2 \text{ o } \varphi \geq 1 \text{ (Since } \lambda_C^2 + \lambda_D^2 \geq 1).$$

$\varphi^{-1}(C) \neq 0\sim$ and $\varphi^{-1}(D) \neq 0\sim$. This contradicts our hypothesis. Hence G_2 is an Pythagorean fuzzy strongly connected space.

Definition-3.16: Let τ be an Pythagorean fuzzy topology on a BCC-algebra G and A be an Pythagorean fuzzy BCC-algebra with Pythagorean fuzzy topology τ_A. Then A is called an Pythagorean fuzzy topological BCC-sub algebra if the self-mapping $\gamma_a: (A, \tau_A) \to (A, \tau_A)$ defined by $\gamma_a(x) = x * a$ for all $a \in G$, is a Relatively Pythagorean fuzzy continuous function.

Theorem-3.17: Let $\varphi: G_1 \to G_2$ be a homomorphism of BCC-algebras and τ and τ^* be Pythagorean fuzzy topologies on G_1 and G_2 respectively such that $\tau = \varphi^{-1}(\tau^*)$. If B is an Pythagorean fuzzy topological BCC-sub algebra in G_2, then $\varphi^{-1}(B)$ is an Pythagorean fuzzy topological BCC-sub algebra in G_1.

Theorem-3.18: Let $\varphi: G_1 \to G_2$ be an isomorphism of BCC-algebras. Let τ and τ^* be the respectively Pythagorean fuzzy topologies on the spaces G_1 and G_2 such that $\tau = \varphi^{-1}(\tau^*)$. If A is an Pythagorean fuzzy topological BCC-sub algebra in G_1, then $\varphi^{-1}(A)$ is an Pythagorean fuzzy topological BCC-sub algebra in G_2.

4. Pythagorean fuzzy topological BCC-ideals

Definition-4.1: An Pythagorean fuzzy set $A = \{\langle \delta_A, \lambda_A \rangle\}$ in a BCK-algebra G is called an Pythagorean fuzzy BCK-ideal of G if the following conditions are satisfied;

(i) $\delta_A^2(0) \geq \delta_A^2(x)$ and $\lambda_A^2(0) \leq \lambda_A^2(x)$,

(ii) $\delta_A^2(x) \geq \min\{\delta_A^2(x*y), \delta_A^2(y)\}$

(iii) $\lambda_A^2(x) \leq \max\{\lambda_A^2(x*y), \lambda_A^2(y)\}$ for all $x, y \in G$.

Definition-4.2: An PythagoreanA $= \langle \delta_A, \lambda_A \rangle$ in G is called an f Pythagorean fuzzy BCC-ideal of G if it satisfies the following conditions;

PF_1: $\delta_A^2(0) \geq \delta_A^2(x)$ and $\lambda_A^2(0) \leq \lambda_A^2(x)$

PF_2: $\delta_A^2(x*z) \geq \text{mi}\{\delta_A^2((x*y)*z), \delta_A^2(y)\}$

PF_3: $\lambda_A^2(x*z) \leq \max\{\lambda_A^2((x*y)*z), \lambda_A^2(y)\}$ for all $x, y, z \in G$.

Putting $z = 0$ in Pythagorean F_2 and Pythagorean F_2, then we can easily see that an Pythagorean fuzzy BCC-ideal is an fermatean fuzzy BCK-ideal. However, the converse does not hold.

Example-4.3: Let $G = \{0, 1, 2, 3, 4, 5\}$ be a BCC-algebra with the following Cayley table;

+	0	1	2	3	4	5
0	0	0	0	0	0	0
1	1	0	0	0	0	1
2	2	2	0	0	1	1
3	3	2	1	0	1	1
4	4	4	4	4	0	1
5	5	5	5	5	5	0

Let $A = \langle \delta_A, \lambda_A \rangle$ be an Pythagorean fuzzy set in G defined by $\delta_A^2(5) = 0.02$, $\delta_A^2(x) = 0.4$, $\lambda_A^2(5) = 0.2$ and $\lambda_A^2(x) = 0.04$ for all $x \neq 5$, then A is an Pythagorean fuzzy BCC-ideal of a BCC-algebra G.

Theorem-4.4: Let φ be a homomorphism of a BCC-algebra G_1 into a BCC-algebra G_2 and B be an Pythagorean fuzzy BCC-ideal of G_2. Then $\varphi^{-1}(B)$ is an Pythagorean fuzzy BCC-ideal of G_1.

Proof: It can be easily seen that

$\delta_{\varphi^{-1}(B)}^2(0) \geq \delta_{\varphi^{-1}(B)}^2(x)$ and $\lambda_{\varphi^{-1}(B)}^2(0) \leq \lambda_{\varphi^{-1}(B)}^2(x)$, for all $x \in G_1$.

For any $x, y, z \in G_1$, we can deduce the following

$$\begin{aligned}\delta_{\varphi^{-1}(B)}^2(x*z) &= \delta_B^2(\varphi(x*z)) \\ &\geq \min\left\{\delta_B^2\left(\varphi((x*y)*z)\right), \delta_B^2(\varphi(y))\right\} \\ &= \min\left\{\delta_B^2\left((\varphi(x)*\varphi(y))*\varphi(z)\right), \delta_B^2(\varphi(y))\right\} \\ &= \min\left\{\delta_{\varphi^{-1}(B)}^2((x*y)*z), \delta_{\varphi^{-1}(B)}^2(y)\right\}.\end{aligned}$$

Also

$$\begin{aligned}\lambda_{\varphi^{-1}(B)}^2(x*z) &= \lambda_B^2(\varphi(x*z)) \\ &\leq \max\left\{\lambda_B^2\left(\varphi((x*y)*z)\right), \lambda_B^2(\varphi(y))\right\} \\ &= \max\left\{\lambda_B^2\left((\varphi(x)*\varphi(y))*\varphi(z)\right), \lambda_B^2(\varphi(y))\right\}\end{aligned}$$

$$= \max\left\{\lambda^2_{\varphi^{-1}(B)}\big((x*y)*z\big), \lambda^2_{\varphi^{-1}(B)}(y)\right\}$$

Hence $\varphi^{-1}(B)$ is an Pythagorean fuzzy BCC-ideal of G_1.

***Corollary-4.5*:** Let φ be a homomorphism of a BCC-algebra G_1 into a BCC-algebra G_2 and B be an Pythagorean fuzzy BCK-ideal of G_2. Then $\varphi^{-1}(B)$ is an Pythagorean fuzzy BCK-ideal of G_1.

Since an Pythagorean BCC-ideal / BCK-ideal is an Pythagorean fuzzy sub algebra, as a consequence of the above results and theorem-3.17, we obtain the following corollary.

***Corollary-4.6*:** Let $\varphi: (G_1, \tau_1) \to (G_2, \tau_2)$ be a homomorphism of the BCC-algebras. Let τ_1 and τ_2 be the Pythagorean fuzzy topologies on G_1 and G_2 respectively such that $\tau_2 = \varphi^{-1}(\tau_1)$. If B is f Pythagorean fuzzy topological BCC-ideal / BCK-ideal of G_2 with the membership function δ^m_B, then $\varphi^{-1}(B)$ is Pythagorean fuzzy topological BCC-ideal / BCK-ideal of G_1 with the membership function $\delta^m_{\varphi^{-1}(B)}$.

***Theorem-4.7*:** Let φ be a homomorphism of a BCC-algebra G_1 into a BCC-algebra G_2. If A is an Pythagorean fuzzy BCC-ideal of G_1, then the homomorphic image φ(A) of A is still an Pythagorean fuzzy BCC-ideal of G_2.

Proof: Let A be an Pythagorean fuzzy topological BCC-ideal of G_1. Then, it is trivial that

$\delta^2_{\varphi(A)}(0) \geq \delta^2_{\varphi(A)}(x)$ and $\lambda^2_{\varphi(A)}(0) \leq \lambda^2_{\varphi(A)}(x)$, for all $x \in G_2$.

Take $x, y, z \in G_2$, and let $x_0 \in \varphi^{-1}(x)$, $y_0 \in \varphi^{-1}(y)$, $z_0 \in \varphi^{-1}(z)$ such that

$\delta^2_A(x_0) = \sup_{t\in \varphi^{-1}(x)} t$, $\delta^2_A(y_0) = \sup_{t\in \varphi^{-1}(y)} t$ and $\delta^2_A(z_0) = \sup_{t\in \varphi^{-1}(z)} t$.

Then we can deduce the following,

$$\delta^2_{\varphi(A)}(x*z) = \sup_{t\in \varphi^{-1}(x*z)} \left(\delta^2_A(t)\right)$$

$$\geq \delta^2_A(x_0 * z_0)$$

$$\geq \min\{\delta^2_A\big((x_0*y_0)*z_0\big), \delta^2_A(y_0)\}$$

$$= \min\left\{\sup_{t\in \varphi^{-1}((x*y)*z)} \left(\delta^2_A(t)\right), \sup_{t\in \varphi^{-1}(y)} \left(\delta^2_A(t)\right)\right\}$$

$$= \min\{\delta^2_{\varphi(A)}\big((x*y)*z\big), \delta^2_{\varphi(A)}(y)\}$$

and $\lambda^2_{\varphi(A)}(x*z) = \inf_{t\in \varphi^{-1}(x*z)} \left(\lambda^2_{\varphi(A)}(t)\right) \leq \lambda^2_A(x_0 * z_0)$

$$\leq \max\{\lambda^2_A\big((x_0*y_0)*z_0\big), \lambda^2_A(y_0)\}$$

$$= \max\left\{\inf_{t\in \varphi^{-1}((x*y)*z)} \left(\lambda^2_A(t)\right), \inf_{t\in \varphi^{-1}(y)} \left(\lambda^2_A(t)\right)\right\}$$

$$= \max\{\lambda^2_{\varphi(A)}\big((x*y)*z\big), \lambda^2_{\varphi(A)}(y)\}$$

Hence $\varphi(A) = \langle \varphi_{\sup}(\delta_A), \varphi_{\inf}(\lambda_A)\rangle$ is induced an Pythagorean fuzzy BCC-ideal of G_2.

Putting $z = 0$ in the above theorem, we obtain:

Corollary-4.8: Let φ be a homomorphism of a BCC-algebra G_1 into a BCC-algebra G_2. If

If A is an Pythagorean fuzzy BCK-ideal of G_1, then the homomorphic image $\varphi(A)$ of A is still an Pythagorean fuzzy BCK-ideal of G_2.

Summing up theorem-3.18, theorem-4.7 and corollary-4.8, we conclude the following theorem.

Theorem-4.9: Let $\varphi: G_1 \to G_2$ be an isomorphism of BCC-algebras. Let τ and τ^* be the respectively Pythagorean fuzzy topologies on the spaces G_1 and G_2 such that $\varphi(\tau) = \tau^*$. If A is an Pythagorean fuzzy topological BCC-ideal / BCK-ideal in G_1, then $\varphi(A)$ is also an Pythagorean fuzzy topological BCC-ideal / BCK-ideal in G_2.

Conclusion: Here we studied the concept of Pythagorean fuzzy topological properties of such algebras such as connectedness, strong connectedness and compact Haussdorff space. We also discussed the characteristic of the homomorphic image and inverse image of Pythagorean fuzzy topological BCC-ideals (BCK-ideals) of BCC-algebras (BCK-algebras).

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