



# THE ASSESSMENT OF VARIOUS NORMAL SPACES IN $\mathfrak{D}_{\mathfrak{R}}$ - NANO TOPOLOGICAL SPACE

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**Abstract :** The main goal of the study is to describe and characterize new classes of normal spaces namely  $\mathfrak{N}\mathfrak{D}_{\mathfrak{R}}\mathfrak{N}\mathfrak{L}$ ,  $\mathfrak{N}_{\mathfrak{S}}\mathfrak{D}_{\mathfrak{R}}\mathfrak{N}\mathfrak{L}$  and  $\mathfrak{N}_{\mathfrak{Q}\mathfrak{S}}\mathfrak{D}_{\mathfrak{R}}\mathfrak{N}\mathfrak{L}$  spaces and to investigate their various properties and features.

## I. INTRODUCTION

In 2013, Lellis Thivagar introduced the concept of Nano-topological spaces [13]. Nano-topology has been analysed by various authors in a diverse array of applications [8-10]. The idea of ideal Nano-topological space was provided by Parimala [6]. In Ideal Nano-topological space, different varieties of generalized open and closed sets are introduced [6, 12]. Maheswari and Rekha have introduced a new kind of ideal Nano-topological space called  $\mathfrak{D}_{\mathfrak{R}}$ -Nano topological space and explored different kinds of open sets in  $\mathfrak{D}_{\mathfrak{R}}$ -Nano topological space [14]. In this current research, we explore the ideas of  $\mathfrak{N}\mathfrak{D}_{\mathfrak{R}}\mathfrak{N}\mathfrak{L}$ ,  $\mathfrak{N}_{\mathfrak{S}}\mathfrak{D}_{\mathfrak{R}}\mathfrak{N}\mathfrak{L}$  and  $\mathfrak{N}_{\mathfrak{Q}\mathfrak{S}}\mathfrak{D}_{\mathfrak{R}}\mathfrak{N}\mathfrak{L}$  -spaces in  $\mathfrak{D}_{\mathfrak{R}}$ -Nano topological space and examine their various characteristics properties.

## 2. Preliminaries

**Definition 2.1**<sup>[14]</sup> Let  $\mathfrak{k}$  be an ideal in the universal set  $\mathfrak{U}$ . Let  $\mathfrak{G} \subseteq \mathfrak{U}$ .

Then the Nano right dynamic lower, upper and boundary regions are given by

$$\underline{\mathfrak{D}}_{\mathfrak{R}}(\mathfrak{G}) = \{\varrho \in \mathfrak{U} : \mathfrak{R}_i(\varrho) \cap \mathfrak{G}^c \in \mathfrak{k} \text{ and } \mathfrak{R}_i(\varrho) \subseteq \mathfrak{G}\},$$

where  $\mathfrak{R}_i(\varrho) = \{\beta \in \mathfrak{U} : \mathfrak{R}_r(\eta) \subseteq \mathfrak{R}_r(\beta)\}$  and

$$\overline{\mathfrak{D}}_{\mathfrak{R}}(\mathfrak{G}) = \{\varrho \in \mathfrak{U} : \mathfrak{R}_i(\varrho) \cap \mathfrak{G} \notin \mathfrak{k} \text{ and } \mathfrak{R}_i \cap \mathfrak{G} \neq \emptyset\} \text{ and } \mathfrak{B}_{\mathfrak{D}_{\mathfrak{R}}}(\mathfrak{G}) = \overline{\mathfrak{D}}_{\mathfrak{R}}(\mathfrak{G}) - \underline{\mathfrak{D}}_{\mathfrak{R}}(\mathfrak{G}).$$

Also, the generated Nano right dynamic topology (called  $\mathfrak{D}_{\mathfrak{R}}$  - Nano topology) with respect  $\mathfrak{G} \subseteq \mathfrak{U}$  is given by  $\mathfrak{N}\mathfrak{D}_{\mathfrak{R}}(\mathfrak{G}) = \{\mathfrak{U}, \emptyset, \overline{\mathfrak{D}}_{\mathfrak{R}}(\mathfrak{G}), \underline{\mathfrak{D}}_{\mathfrak{R}}(\mathfrak{G}), \mathfrak{B}_{\mathfrak{D}_{\mathfrak{R}}}(\mathfrak{G})\}$ .

The collection  $\mathfrak{B}_{\mathfrak{D}_{\mathfrak{R}}}(\mathfrak{G}) = \{\mathfrak{U}, \underline{\mathfrak{D}}_{\mathfrak{R}}(\mathfrak{G}), \mathfrak{B}_{\mathfrak{D}_{\mathfrak{R}}}(\mathfrak{G})\}$  forms a basis for  $\mathfrak{N}\mathfrak{D}_{\mathfrak{R}}$  and the Nano right dynamic topological space is denoted by  $(\mathfrak{U}, \mathfrak{N}\mathfrak{D}_{\mathfrak{R}}(\mathfrak{G}), \mathfrak{k})$ . The elements of  $\mathfrak{N}\mathfrak{D}_{\mathfrak{R}}(\mathfrak{G})$  are called  $\mathfrak{N}\mathfrak{D}_{\mathfrak{R}}$ - open sets and its complements are called  $\mathfrak{N}\mathfrak{D}_{\mathfrak{R}}$ - closed sets.

**Definition 2.2**<sup>[14]</sup> Let  $(\mathfrak{U}, \mathfrak{N}\mathfrak{D}_{\mathfrak{R}}(\mathfrak{G}), \mathfrak{k})$  be a nano right dynamic topological space. An operator  $\mathfrak{Q}^{\mathfrak{D}_{\mathfrak{R}}} : \mathfrak{p}(\mathfrak{U}) \rightarrow \mathfrak{p}(\mathfrak{U})$  is known as the dynamic function of  $\mathfrak{k}$  on  $\mathfrak{U}$  and defined as  $\mathfrak{Q}^{\mathfrak{D}_{\mathfrak{R}}} = \{\varrho \in \mathfrak{U} : \mathfrak{U} \cap \varrho \notin \mathfrak{k} \text{ for all } \mathfrak{U} \in \mathfrak{N}\mathfrak{D}_{\mathfrak{R}}(\mathfrak{G})\}$  and its closure operator is  $\mathfrak{D}_{\mathfrak{R}}^*(\mathfrak{Q}) = \mathfrak{Q} \cup \mathfrak{Q}^{\mathfrak{D}_{\mathfrak{R}}}$ .

**Definition 2.2**<sup>[14]</sup> A subset  $\mathfrak{P}$  of  $(\mathfrak{U}, \mathfrak{N}\mathfrak{D}_{\mathfrak{R}}(\mathfrak{G}), \mathfrak{k})$  is  $\mathfrak{N}_{\mathfrak{r}}\mathfrak{D}_{\mathfrak{R}}$ - open if  $\mathfrak{P} = \text{int}_{\mathfrak{N}\mathfrak{D}_{\mathfrak{R}}}(\text{cl}_{\mathfrak{N}\mathfrak{D}_{\mathfrak{R}}}(\mathfrak{P}))$

**Definition 2.3**<sup>[14]</sup> Let  $H$  be a subset of  $(\mathfrak{U}, \mathfrak{N}\mathfrak{D}_{\mathfrak{R}}(\mathfrak{G}), \mathfrak{k})$ . Then,

$H$  is  $\mathfrak{N}_{\mathfrak{S}}\mathfrak{D}_{\mathfrak{R}}$ -closed if  $\mathfrak{D}_{\mathfrak{R}}^*(H) \subseteq S$  whenever  $H \subseteq S$ ,  $S$  is  $\mathfrak{N}_{\mathfrak{S}}\mathfrak{D}_{\mathfrak{R}}$ - open.

**Example 2.4.** Let  $\mathfrak{U} = \{\rho, \varsigma, v, \xi, \tau\}$ ,  $\mathfrak{G} = \{\rho, \varsigma\}$ ,  $\mathfrak{k} = \{\emptyset, \{\rho\}, \{\xi\}, \{\rho, \xi\}\}$  and the relation  $X = \{(\rho, \rho), (\rho, \varsigma), (\rho, v), (\varsigma, \varsigma), (v, \rho), (v, \xi), (\xi, \tau), (\tau, \varsigma)\}$ .

Then the corresponding  $\mathfrak{D}_{\mathfrak{R}}$  - Nano topology is

$$\mathfrak{N}\mathfrak{D}_{\mathfrak{R}}(\mathfrak{M}) = \{\mathfrak{U}, \emptyset, \{\rho\}, \{\rho, \varsigma, \tau\}, \{\varsigma, \tau\}\}.$$

Then  $\mathfrak{N}\mathfrak{D}_{\mathfrak{R}}\mathfrak{C}(\mathfrak{C}) = \{\mathfrak{U}, \emptyset, \{v, \xi\}, \{\rho, v, \xi\}, \{\varsigma, v, \xi, \tau\}\}$ ,

$$\mathfrak{D}_{\mathfrak{R}}^*\mathfrak{C}(\mathfrak{C}) = \{\mathfrak{U}, \emptyset, \{\rho\}, \{\xi\}, \{\rho, \xi\}, \{v, \xi\}, \{\rho, v, \xi\}, \{\varsigma, v, \xi, \tau\}\}$$

$$\mathfrak{N}_{\mathfrak{S}}\mathfrak{D}_{\mathfrak{R}}\mathfrak{C}(\mathfrak{G}) = \{\mathfrak{U}, \emptyset, \{\rho\}, \{\xi\}, \{\rho, \xi\}, \{c, \xi\}, \{\rho, v, \xi\}, \{\varsigma, v, \xi\}, \{v, \xi, \tau\}, \{\rho, \varsigma, v, \xi\}, \{\rho, v, \xi, \tau\}, \{\varsigma, v, \xi, \tau\}\}.$$

### 3. $\mathfrak{N}_{\mathcal{D}_{\mathfrak{R}}}$ $\mathfrak{N}\ell$ -spaces

**Definition 3.1.** Let  $(\mathcal{U}, \mathfrak{N}_{\mathcal{D}_{\mathfrak{R}}}(\mathcal{P}), \mathcal{K})$  be a  $\mathfrak{N}_{\mathcal{D}_{\mathfrak{R}}}$ - Nano topological space. Then,

1.  $\mathcal{U}$  is  $\mathfrak{N}_{\mathcal{D}_{\mathfrak{R}}}$ -Normal space ( known to be  $\mathfrak{N}_{\mathcal{D}_{\mathfrak{R}}}$  $\mathfrak{N}\ell$ -space ) if with any pair of non-empty disjoint  $\mathfrak{N}_{\mathcal{D}_{\mathfrak{R}}}$ - closed sets  $\mathcal{S}_f$  and  $\mathcal{S}_h$  of  $(\mathcal{U}, \mathfrak{N}_{\mathcal{D}_{\mathfrak{R}}}(\mathcal{P}), \mathcal{K})$ , there exist  $\mathfrak{N}_{\mathcal{D}_{\mathfrak{R}}}$ -open sets  $\mathcal{G}_f$  and  $\mathcal{G}_h$  of  $(\mathcal{U}, \mathfrak{N}_{\mathcal{D}_{\mathfrak{R}}}(\mathcal{P}), \mathcal{K})$  with  $\mathcal{S}_f \subset \mathcal{G}_f$  and  $\mathcal{S}_h \subset \mathcal{G}_h$ .
2.  $\mathcal{U}$  is  $\mathfrak{N}_{\mathcal{D}_{\mathfrak{R}}}$ -Normal space( known to be  $\mathfrak{N}_{\mathcal{D}_{\mathfrak{R}}}$  $\mathfrak{N}\ell$ -space ) if with any pair of non-empty disjoint  $\mathfrak{N}_{\mathcal{D}_{\mathfrak{R}}}$ - closed sets  $\mathcal{S}_f$  and  $\mathcal{S}_\psi$  of  $(\mathcal{U}, \mathfrak{N}_{\mathcal{D}_{\mathfrak{R}}}(\mathcal{P}), \mathcal{K})$ , there exist  $\mathfrak{N}_{\mathcal{D}_{\mathfrak{R}}}$ -open sets  $\mathcal{G}_f$  and  $\mathcal{G}_h$  of  $(\mathcal{U}, \mathfrak{N}_{\mathcal{D}_{\mathfrak{R}}}(\mathcal{P}), \mathcal{K})$  with  $\mathcal{S}_f \subset \mathcal{G}_f$  and  $\mathcal{S}_h \subset \mathcal{G}_h$ .

**Theorem 3.2.** Each  $\mathfrak{N}_{\mathcal{D}_{\mathfrak{R}}}$  $\mathfrak{N}\ell$ -space is  $\mathfrak{N}_{\mathcal{D}_{\mathfrak{R}}}$  $\mathfrak{N}\ell$ -space.

**Proof.** Let  $(\mathcal{U}, \mathfrak{N}_{\mathcal{D}_{\mathfrak{R}}}(\mathcal{P}), \mathcal{K})$  be a  $\mathfrak{N}_{\mathcal{D}_{\mathfrak{R}}}$  $\mathfrak{N}\ell$ -space and  $\mathcal{S}_f$  and  $\mathcal{S}_h$  be two disjoint non-empty  $\mathfrak{N}_{\mathcal{D}_{\mathfrak{R}}}$ - closed sets. Then there exist disjoint non – empty pair of  $\mathfrak{N}_{\mathcal{D}_{\mathfrak{R}}}$ -open sets  $\mathcal{G}_f$  and  $\mathcal{G}_h$  of  $(\mathcal{U}, \mathfrak{N}_{\mathcal{D}_{\mathfrak{R}}}(\mathcal{P}), \mathcal{K})$  with  $\mathcal{S}_f \subset \mathcal{G}_f$  and  $\mathcal{S}_h \subset \mathcal{G}_h$ . Here  $\mathcal{G}_f$  and  $\mathcal{G}_h$  are  $\mathfrak{N}_{\mathcal{D}_{\mathfrak{R}}}$ -open sets.

Since each  $\mathfrak{N}_{\mathcal{D}_{\mathfrak{R}}}$ - open set is an  $\mathfrak{N}_{\mathcal{D}_{\mathfrak{R}}}$ -open set,  $(\mathcal{U}, \mathfrak{N}_{\mathcal{D}_{\mathfrak{R}}}(\mathcal{P}), \mathcal{K})$  is  $\mathfrak{N}_{\mathcal{D}_{\mathfrak{R}}}$  $\mathfrak{N}\ell$ -space.

**Remark 3.3.** From example 2.4,  $\{\rho, \varsigma\}$  is  $\mathfrak{N}_{\mathcal{D}_{\mathfrak{R}}}$ -open but not  $\mathfrak{N}_{\mathcal{D}_{\mathfrak{R}}}$ -open.

Therefore every  $\mathfrak{N}_{\mathcal{D}_{\mathfrak{R}}}$  $\mathfrak{N}\ell$ -space need not be a  $\mathfrak{N}_{\mathcal{D}_{\mathfrak{R}}}$  $\mathfrak{N}\ell$ -space.

**Theorem 3.4.** If  $(\mathcal{U}, \mathfrak{N}_{\mathcal{D}_{\mathfrak{R}}}(\mathcal{P}), \mathcal{K})$  is  $\mathfrak{N}_{\mathcal{D}_{\mathfrak{R}}}$  $\mathfrak{N}\ell$ -space then for each pair of  $\mathfrak{N}_{\mathcal{D}_{\mathfrak{R}}}$ -open sets  $\mathcal{S}_f$  and  $\mathcal{S}_h$  with  $\mathcal{S}_f \cap \mathcal{S}_h = \mathcal{U}$ , there exist  $\mathfrak{N}_{\mathcal{D}_{\mathfrak{R}}}$ -closed sets  $\mathcal{G}_f$  and  $\mathcal{G}_h$  with  $\mathcal{G}_f \subset \mathcal{S}_f$  and  $\mathcal{G}_h \subset \mathcal{S}_h$  and  $\mathcal{G}_f \cap \mathcal{G}_h = \mathcal{U}$ .

**Proof.** Let  $\mathcal{S}_f$  and  $\mathcal{S}_h$  be  $\mathfrak{N}_{\mathcal{D}_{\mathfrak{R}}}$ -open sets in  $(\mathcal{U}, \mathfrak{N}_{\mathcal{D}_{\mathfrak{R}}}(\mathcal{P}), \mathcal{K})$  with  $\mathcal{S}_f \cap \mathcal{S}_h = \mathcal{U}$ .

Then  $\mathcal{S}_f^c$  and  $\mathcal{S}_h^c$  are disjoint  $\mathfrak{N}_{\mathcal{D}_{\mathfrak{R}}}$ - closed sets. Since  $(\mathcal{U}, \mathfrak{N}_{\mathcal{D}_{\mathfrak{R}}}(\mathcal{P}), \mathcal{K})$  is  $\mathfrak{N}_{\mathcal{D}_{\mathfrak{R}}}$  $\mathfrak{N}\ell$ -space, their exist pair of  $\mathfrak{N}_{\mathcal{D}_{\mathfrak{R}}}$ -open sets  $\mathcal{X}_f$  and  $\mathcal{X}_h$  such that  $\mathcal{S}_f^c \subset \mathcal{X}_f$  and  $\mathcal{S}_h^c \subset \mathcal{X}_h$ .

This shows that  $\mathcal{G}_f = \mathcal{S}_f^c$  and  $\mathcal{G}_h = \mathcal{S}_h^c$  are  $\mathfrak{N}_{\mathcal{D}_{\mathfrak{R}}}$ -closed sets such that  $\mathcal{G}_f \subset \mathcal{S}_f$  and  $\mathcal{G}_h \subset \mathcal{S}_h$  with  $\mathcal{G}_f \cap \mathcal{G}_h = \mathcal{U}$ .

**Theorem 3.5.** Let  $\Lambda: (\mathcal{U}_m, \mathfrak{N}^m, \mathcal{K}_m) \rightarrow (\mathcal{U}_n, \mathfrak{N}^n, \mathcal{K}_n)$  be a  $\mathfrak{N}_{\mathcal{D}_{\mathfrak{R}}}$ - continuous bijective function and  $\mathfrak{N}_{\mathcal{D}_{\mathfrak{R}}}$ -open function. If  $\mathcal{U}_m$  is a  $\mathfrak{N}_{\mathcal{D}_{\mathfrak{R}}}$  $\mathfrak{N}\ell$ -space, then  $\mathcal{U}_n$  is a  $\mathfrak{N}_{\mathcal{D}_{\mathfrak{R}}}$  $\mathfrak{N}\ell$ -space.

**Proof.** Let  $\mathcal{S}_f$  and  $\mathcal{S}_h$  be any pair of disjoint  $\mathfrak{N}_{\mathcal{D}_{\mathfrak{R}}}$ - closed sets in  $(\mathcal{U}_n, \mathfrak{N}^n, \mathcal{K}_n)$ .

As  $\Lambda$  is  $\mathfrak{N}_{\mathcal{D}_{\mathfrak{R}}}$ -continuous bijective,  $\Lambda^{-1}(\mathcal{S}_f)$  and  $\Lambda^{-1}(\mathcal{S}_h)$  are disjoint  $\mathfrak{N}_{\mathcal{D}_{\mathfrak{R}}}$ - closed sets in  $(\mathcal{U}_m, \mathfrak{N}^m, \mathcal{K}_m)$ . Also since  $\mathcal{U}_m$  is  $\mathfrak{N}_{\mathcal{D}_{\mathfrak{R}}}$  $\mathfrak{N}\ell$ -space, there exist disjoint  $\mathfrak{N}_{\mathcal{D}_{\mathfrak{R}}}$ -open sets  $\mathcal{S}_\eta$  and  $\mathcal{S}_h$  with  $\Lambda^{-1}(\mathcal{S}_\eta) \subset \mathcal{S}_\eta$  and  $\Lambda^{-1}(\mathcal{S}_h) \subset \mathcal{S}_h$ . Therefore,  $\mathcal{S}_f \subset \Lambda(\mathcal{S}_\eta)$  and  $\mathcal{S}_h \subset \Lambda(\mathcal{S}_h)$ .

Also,  $\Lambda(\mathcal{S}_f)$  and  $\Lambda(\mathcal{S}_h)$  are  $\mathfrak{N}_{\mathcal{D}_{\mathfrak{R}}}$ -open sets in  $(\mathcal{U}_n, \mathfrak{N}^n, \mathcal{K}_n)$ . Since  $\Lambda$  is bijective,

$$\Lambda(\mathcal{G}_f) \cap \Lambda(\mathcal{S}_h) = \emptyset. \text{ Hence, } (\mathcal{U}_n, \mathfrak{N}^n, \mathcal{K}_n) \text{ is } \mathfrak{N}_{\mathcal{D}_{\mathfrak{R}}}\mathfrak{N}\ell\text{-space.}$$

**Theorem 3.6.** Let  $\Lambda: (\mathcal{U}_m, \mathfrak{N}^m, \mathcal{K}_m) \rightarrow (\mathcal{U}_n, \mathfrak{N}^n, \mathcal{K}_n)$  be a  $\mathfrak{N}_{\mathcal{D}_{\mathfrak{R}}}$ -continuous,  $\mathfrak{N}_{\mathcal{D}_{\mathfrak{R}}}$ -closed bijection and  $\mathcal{U}_n$  be a  $\mathfrak{N}_{\mathcal{D}_{\mathfrak{R}}}$  $\mathfrak{N}\ell$ -space. Then  $\mathcal{U}_m$  is a  $\mathfrak{N}_{\mathcal{D}_{\mathfrak{R}}}$  $\mathfrak{N}\ell$ -space.

**Proof.** Let  $\mathcal{S}_f$  and  $\mathcal{S}_\psi$  be any pair of disjoint  $\mathfrak{N}_{\mathcal{D}_{\mathfrak{R}}}$ - closed sets in  $(\mathcal{U}_m, \mathfrak{N}^m, \mathcal{K}_m)$ .

Then  $\Lambda(\mathcal{S}_f)$  and  $\Lambda(\mathcal{S}_\psi)$  are disjoint closed sets in  $(\mathcal{U}_n, \mathfrak{N}^n, \mathcal{K}_n)$ .

Since  $\mathcal{U}_n$  is  $\mathfrak{N}_{\mathcal{D}_{\mathfrak{R}}}$  $\mathfrak{N}\ell$ -space, there exist disjoint  $\mathfrak{N}_{\mathcal{D}_{\mathfrak{R}}}$ -open sets  $\mathcal{G}_f$  and  $\mathcal{G}_\psi$  with  $\Lambda(\mathcal{S}_f) \subset \mathcal{G}_f$

and  $\Lambda(\mathcal{S}_\psi) \subset \mathcal{G}_\psi$ . Therefore  $\mathcal{S}_f \subset \Lambda^{-1}(\mathcal{G}_f)$  and  $\mathcal{S}_\psi \subset \Lambda^{-1}(\mathcal{G}_\psi)$ . Since  $\Lambda$  is  $\mathfrak{N}_{\mathcal{D}_{\mathfrak{R}}}$ -continuous,  $\Lambda^{-1}(\mathcal{G}_f)$  and  $\Lambda^{-1}(\mathcal{G}_\psi)$  are  $\mathfrak{N}_{\mathcal{D}_{\mathfrak{R}}}$ -open sets in  $(\mathcal{U}_m, \mathfrak{N}^m, \mathcal{K}_m)$ .

Also,  $\Lambda^{-1}(\mathcal{G}_f) \cap \Lambda^{-1}(\mathcal{G}_\psi) = \emptyset$ . Hence  $(\mathcal{U}_m, \mathfrak{N}^m, \mathcal{K}_m)$  is  $\mathfrak{N}_{\mathcal{D}_{\mathfrak{R}}}$  $\mathfrak{N}\ell$ -space.

**Theorem 3.7.** If  $\Lambda: (\mathcal{U}_m, \mathfrak{N}^m, \mathcal{K}_m) \rightarrow (\mathcal{U}_n, \mathfrak{N}^n, \mathcal{K}_n)$  is  $\mathfrak{N}_{\mathcal{D}_{\mathfrak{R}}}$ -irresolute,  $\mathfrak{N}_{\mathcal{D}_{\mathfrak{R}}}$ -closed bijection and  $\mathcal{U}_n$  is  $\mathfrak{N}_{\mathcal{D}_{\mathfrak{R}}}$  $\mathfrak{N}\ell$ -space, then  $\mathcal{U}_m$  is a  $\mathfrak{N}_{\mathcal{D}_{\mathfrak{R}}}$  $\mathfrak{N}\ell$ -space.

**Proof.** Let  $\mathcal{S}_f$  and  $\mathcal{S}_h$  be any pair of disjoint  $\mathfrak{N}_{\mathcal{D}_{\mathfrak{R}}}$ - closed sets in  $(\mathcal{U}_m, \mathfrak{N}^m, \mathcal{K}_m)$ .

As  $\Lambda$  is  $\mathfrak{N}_{\mathcal{D}_{\mathfrak{R}}}$ -closed bijection,  $\Lambda(\mathcal{S}_f)$  and  $\Lambda(\mathcal{S}_h)$  are disjoint  $\mathfrak{N}_{\mathcal{D}_{\mathfrak{R}}}$ - closed sets in  $(\mathcal{U}_n, \mathfrak{N}^n, \mathcal{K}_n)$ .

Also since  $(\mathcal{U}_n, \mathfrak{N}^n, \mathcal{K}_n)$  is  $\mathfrak{N}_{\mathcal{D}_{\mathfrak{R}}}$  $\mathfrak{N}\ell$ -space, there exist disjoint  $\mathfrak{N}_{\mathcal{D}_{\mathfrak{R}}}$ -open sets in  $\mathcal{G}_f$  and  $\mathcal{G}_h$  such that  $\Lambda(\mathcal{S}_f) \subset \mathcal{G}_f$  and  $\Lambda(\mathcal{S}_h) \subset \mathcal{G}_h$ .

Here  $\Lambda$  is  $\mathfrak{N}_{\mathcal{D}_{\mathfrak{R}}}$ -irresolute,  $\Lambda^{-1}(\mathcal{G}_f)$  and  $\Lambda^{-1}(\mathcal{G}_h)$  are  $\mathfrak{N}_{\mathcal{D}_{\mathfrak{R}}}$ -open sets

in  $(\mathcal{U}_m, \mathfrak{N}^m, \mathcal{K}_m)$  and also  $\Lambda^{-1}(\mathcal{G}_f) \cap \Lambda^{-1}(\mathcal{G}_h) = \emptyset$ .

Hence  $(\mathcal{U}_m, \mathfrak{N}^m, \mathcal{K}_m)$  is  $\mathfrak{N}_{\mathcal{D}_{\mathfrak{R}}}$  $\mathfrak{N}\ell$ -space.

### 4. $\mathfrak{N}_{\mathcal{D}_{\mathfrak{R}}}$ $\mathfrak{N}\ell$ -spaces

**Definition 4.1.** A subset  $\mathcal{D}_\zeta$  of  $(\mathcal{U}, \mathfrak{N}_{\mathcal{D}_{\mathfrak{R}}}(G), \mathcal{K})$  is a  $\mathfrak{N}_{\mathcal{D}_{\mathfrak{R}}}$ -open set if it is the finite union of  $\mathfrak{N}_{\mathcal{D}_{\mathfrak{R}}}$ -open sets.

**Example 4.2.** Let  $\mathcal{U} = \{\rho, \varsigma, v, \xi, \tau\}$ ,  $\mathcal{G} = \{\rho, \varsigma\}$ ,  $\mathcal{K} = \{\emptyset, \{\rho\}, \{\xi\}, \{\rho, \xi\}\}$  and the relation  $X = \{(\rho, \rho), (\rho, \varsigma), (\rho, v), (\varsigma, \varsigma), (v, \rho), (v, \xi), (\xi, \tau), (\tau, \varsigma)\}$ .

Then the corresponding  $\mathfrak{N}_{\mathcal{D}_{\mathfrak{R}}}$ - Nano topology is

$\mathfrak{N}_{\mathcal{D}_{\mathfrak{R}}}(\mathcal{M}) = \{\mathcal{U}, \emptyset, \{\rho\}, \{\rho, \varsigma, \tau\}, \{\varsigma, \tau\}\}$ . Also,  $\{\rho, \varsigma, \tau\}$  is a  $\mathfrak{N}_{\mathcal{D}_{\mathfrak{R}}}$ -open set.

**Definition 4.3.** A  $\mathfrak{N}_{\mathcal{D}_{\mathfrak{R}}}$ -nano topological space  $(\mathcal{U}, \mathfrak{N}_{\mathcal{D}_{\mathfrak{R}}}(G), \mathcal{K})$  is  $\mathfrak{N}_{\mathcal{D}_{\mathfrak{R}}}$  $\mathfrak{N}\ell$ -space if for any pair of disjoint  $\mathfrak{N}_{\mathcal{D}_{\mathfrak{R}}}$ -closed sub sets  $\mathcal{W}_\eta$ ,  $\mathcal{W}_\psi$  of  $\mathcal{U}$ , there exist disjoint  $\mathfrak{N}_{\mathcal{D}_{\mathfrak{R}}}$ -open sets  $\mathcal{V}_\eta$ ,  $\mathcal{V}_\psi$  such that  $\mathcal{W}_\eta \subseteq \mathcal{V}_\eta$  and  $\mathcal{W}_\psi \subseteq \mathcal{V}_\psi$ .

**Theorem 4.4.** If  $(\mathcal{U}, \mathfrak{N}_{\mathcal{D}_{\mathfrak{R}}}(G), \mathcal{K})$  is a  $\mathfrak{N}_{\mathcal{D}_{\mathfrak{R}}}$ -nano topological space, then the following are equivalent.

- a)  $\mathcal{U}$  is  $\mathfrak{N}_{\overline{\mathcal{D}}\mathfrak{D}}\mathfrak{N}\ell$ -space.
- b) For any pair of  $\mathfrak{N}_{\overline{\mathcal{D}}\mathfrak{D}}$ -open sets  $\mathfrak{W}_\eta$  and  $\mathfrak{W}_\psi$  of  $\mathcal{U}$  with  $\mathfrak{W}_\eta \cup \mathfrak{W}_\psi = \mathcal{U}$ , there are  $\mathfrak{N}_{\overline{\mathcal{D}}\mathfrak{D}}$ -closed subsets  $\mathfrak{Y}_\eta, \mathfrak{Y}_\psi$  of  $\mathcal{U}$  such that  $\mathfrak{Y}_\eta \subseteq \mathfrak{W}_\eta$  and  $\mathfrak{Y}_\psi \subseteq \mathfrak{W}_\psi$  with  $\mathfrak{Y}_\eta \cup \mathfrak{Y}_\psi = \mathcal{U}$ .
- c) For every  $\mathfrak{N}_{\overline{\mathcal{D}}\mathfrak{D}}$ -closed set  $\mathfrak{W}_\eta$  and every  $\mathfrak{N}_{\overline{\mathcal{D}}\mathfrak{D}}$ -open set  $\mathfrak{Y}_\eta$  with  $\mathfrak{W}_\eta \subseteq \mathfrak{Y}_\eta$ , there is  $\mathfrak{C}_\eta$  with  $\mathfrak{W}_\eta \subseteq \mathfrak{C}_\eta \subseteq \mathfrak{N}_{\overline{\mathcal{D}}\mathfrak{D}}\mathfrak{C}(\mathfrak{C}_\eta) \subseteq \mathfrak{Y}_\eta$ .
- d) For any pair of disjoint  $\mathfrak{N}_{\overline{\mathcal{D}}\mathfrak{D}}$ -closed subsets  $\mathfrak{W}_\eta$  and  $\mathfrak{W}_\psi$  of  $\mathcal{U}$ , there exist  $\mathfrak{N}_{\overline{\mathcal{D}}\mathfrak{D}}$ -open subsets  $\mathfrak{C}_\eta$  and  $\mathfrak{C}_\psi$  of  $\mathcal{U}$  with  $\mathfrak{W}_\eta \subseteq \mathfrak{C}_\eta, \mathfrak{W}_\psi \subseteq \mathfrak{C}_\psi$  and  $\mathfrak{N}_{\overline{\mathcal{D}}\mathfrak{D}}\mathfrak{C}(\mathfrak{C}_\eta) \cap \mathfrak{N}_{\overline{\mathcal{D}}\mathfrak{D}}\mathfrak{C}(\mathfrak{C}_\psi) = \emptyset$ .

**Proof.** Assume (a). Let  $\mathfrak{W}_\eta$  and  $\mathfrak{W}_\psi$  be  $\mathfrak{N}_{\overline{\mathcal{D}}\mathfrak{D}}$ -open subsets of  $\mathfrak{N}_{\overline{\mathcal{D}}\mathfrak{D}}\mathfrak{N}\ell$ -space  $\mathcal{U}$  with  $\mathfrak{W}_\eta \cup \mathfrak{W}_\psi = \mathcal{U}$ . Then  $\mathfrak{W}_\eta^c$  and  $\mathfrak{W}_\psi^c$  are the  $\mathfrak{N}_{\overline{\mathcal{D}}\mathfrak{D}}$ -closed subsets of  $\mathcal{U}$ .

Also, there exist disjoint  $\mathfrak{N}_{\overline{\mathcal{D}}\mathfrak{D}}$ -open subsets  $\mathfrak{L}_\eta, \mathfrak{L}_\psi$  of  $\mathcal{U}$  with  $\mathfrak{W}_\eta^c \subseteq \mathfrak{L}_\eta$

and  $\mathfrak{W}_\psi^c \subseteq \mathfrak{L}_\psi$ . Let  $\mathfrak{Y}_\eta = \mathfrak{L}_\eta^c$  and  $\mathfrak{Y}_\psi = \mathfrak{L}_\psi^c$ . Then  $\mathfrak{L}_\eta$  and  $\mathfrak{L}_\psi$  are  $\mathfrak{N}_{\overline{\mathcal{D}}\mathfrak{D}}$ -closed subsets of  $\mathcal{U}$  with  $\mathfrak{L}_\eta \subseteq \mathfrak{W}_\eta$  and  $\mathfrak{L}_\psi \subseteq \mathfrak{W}_\psi$ . Hence (b) is proved. Assume (b). Let  $\mathfrak{W}_\eta$  be any  $\mathfrak{N}_{\overline{\mathcal{D}}\mathfrak{D}}$ -closed set and  $\mathfrak{C}_\eta$  be any  $\mathfrak{N}_{\overline{\mathcal{D}}\mathfrak{D}}$ -open subset of  $\mathcal{U}$  with  $\mathfrak{W}_\eta \subseteq \mathfrak{C}_\eta$ . Then  $\mathfrak{W}_\eta^c$  and  $\mathfrak{C}_\eta$  are  $\mathfrak{N}_{\overline{\mathcal{D}}\mathfrak{D}}$ -open subsets of  $\mathcal{U}$  with  $\mathfrak{W}_\eta^c \cup \mathfrak{C}_\eta = \mathcal{U}$ . Then there exist  $\mathfrak{N}_{\overline{\mathcal{D}}\mathfrak{D}}$ -closed subsets of  $\mathcal{U}$   $\mathfrak{L}_\eta, \mathfrak{L}_\psi$  of  $\mathcal{U}$  with  $\mathfrak{L}_\eta \subseteq \mathfrak{W}_\eta^c$  and  $\mathfrak{L}_\psi \subseteq \mathfrak{C}_\eta$  and  $\mathfrak{L}_\eta \cup \mathfrak{L}_\psi = \mathcal{U}$ . Also,  $\mathfrak{W}_\eta \subseteq \mathfrak{L}_\eta^c, \mathfrak{C}_\eta \subseteq \mathfrak{L}_\psi^c$  and  $\mathfrak{L}_\eta^c \cap \mathfrak{L}_\psi^c = \emptyset$ . Let  $\mathfrak{C}_\eta = \mathfrak{L}_\eta^c$  and  $\mathfrak{C}_\psi = \mathfrak{L}_\psi^c$ . Then  $\mathfrak{C}_\eta$  and  $\mathfrak{C}_\psi$  are disjoint  $\mathfrak{N}_{\overline{\mathcal{D}}\mathfrak{D}}$ -open sets with  $\mathfrak{W}_\eta \subseteq \mathfrak{C}_\eta$  and  $\mathfrak{C}_\psi \subseteq \mathfrak{Y}_\psi$  which gives  $\mathfrak{C}_\psi^c \subseteq \mathfrak{Y}_\psi$ .

Since  $\mathfrak{C}_\psi^c$  is  $\mathfrak{N}_{\overline{\mathcal{D}}\mathfrak{D}}$ -closed,  $\mathfrak{N}_{\overline{\mathcal{D}}\mathfrak{D}}\mathfrak{C}(\mathfrak{C}_\eta) \subseteq \mathfrak{Y}_\eta$ . Hence (c) is proved.

Assume (c). Let  $\mathfrak{W}_\eta$  and  $\mathfrak{W}_\psi$  be any pair of disjoint  $\mathfrak{N}_{\overline{\mathcal{D}}\mathfrak{D}}$ -closed sets of  $\mathcal{U}$ .

Then  $\mathfrak{W}_\eta \subseteq \mathfrak{W}_\eta^c$  and  $\mathfrak{W}_\psi^c$  is  $\mathfrak{N}_{\overline{\mathcal{D}}\mathfrak{D}}$ -open. After that, there is a  $\mathfrak{N}_{\overline{\mathcal{D}}\mathfrak{D}}$ -open sets  $\mathfrak{Y}_\eta$  of  $\mathcal{U}$  with  $\mathfrak{W}_\eta \subseteq \mathfrak{Y}_\eta \subseteq \mathfrak{N}_{\overline{\mathcal{D}}\mathfrak{D}}\mathfrak{C}(\mathfrak{Y}_\eta) \subseteq \mathfrak{W}_\eta^c$ . Also  $\mathfrak{W}_\eta \subseteq \mathfrak{Y}_\eta$ , and  $\mathfrak{W}_\psi \subseteq (\mathfrak{N}_{\overline{\mathcal{D}}\mathfrak{D}}\mathfrak{C}(\mathfrak{Y}_\eta))^c$ .

Let  $\mathfrak{Y}_\psi = (\mathfrak{N}_{\overline{\mathcal{D}}\mathfrak{D}}\mathfrak{C}(\mathfrak{Y}_\eta))^c$ . Then  $\mathfrak{Y}_\psi$  is  $\mathfrak{N}_{\overline{\mathcal{D}}\mathfrak{D}}$ -open subset of  $\mathcal{U}$ .

Therefore,  $\mathfrak{W}_\eta \subseteq \mathfrak{Y}_\eta$  and  $\mathfrak{W}_\psi \subseteq \mathfrak{Y}_\psi$  with  $\mathfrak{N}_{\overline{\mathcal{D}}\mathfrak{D}}\mathfrak{C}(\mathfrak{Y}_\eta) \cap \mathfrak{N}_{\overline{\mathcal{D}}\mathfrak{D}}\mathfrak{C}(\mathfrak{Y}_\psi) = \emptyset$ .

Hence (d) is proved.

From the definition of  $\mathfrak{N}_{\overline{\mathcal{D}}\mathfrak{D}}\mathfrak{N}\ell$ -space, (d) $\Rightarrow$ (a) can be easily verified.

**Theorem 4.5.** If  $\Lambda: (\mathcal{U}_a, \mathfrak{N}^a, \mathfrak{K}_a) \rightarrow (\mathcal{U}_b, \mathfrak{N}^b, \mathfrak{K}_b)$  is  $\mathfrak{N}_{\overline{\mathcal{D}}\mathfrak{D}}$ -open and  $\mathfrak{N}_{\overline{\mathcal{D}}\mathfrak{D}}$ -continuous bijective function and  $(\mathcal{U}_a, \mathfrak{N}^a, \mathfrak{K}_a)$  is  $\mathfrak{N}_{\overline{\mathcal{D}}\mathfrak{D}}\mathfrak{N}\ell$ -space, then  $\Lambda(\mathcal{U}_a)$  is  $\mathfrak{N}_{\overline{\mathcal{D}}\mathfrak{D}}\mathfrak{N}\ell$ -space.

**Proof.** Let  $\mathfrak{L}_\eta$  and  $\mathfrak{L}_\psi$  be two disjoint  $\mathfrak{N}_{\overline{\mathcal{D}}\mathfrak{D}}$ -closed subsets of  $\Lambda(\mathcal{U}_a)$ . Then,  $\Lambda^{-1}(\mathfrak{L}_\eta)$  and  $\Lambda^{-1}(\mathfrak{L}_\psi)$  are disjoint  $\mathfrak{N}_{\overline{\mathcal{D}}\mathfrak{D}}$ -closed subsets of  $\mathcal{U}_a$ . Since  $\mathcal{U}_a$  is a  $\mathfrak{N}_{\overline{\mathcal{D}}\mathfrak{D}}\mathfrak{N}\ell$ -space, there exist  $\mathfrak{N}_{\overline{\mathcal{D}}\mathfrak{D}}$ -open subsets  $\mathfrak{Y}_\eta$  and  $\mathfrak{Y}_\psi$  of  $\mathcal{U}_a$  with  $\Lambda^{-1}(\mathfrak{L}_\eta) \subseteq \mathfrak{Y}_\eta, \Lambda^{-1}(\mathfrak{L}_\psi) \subseteq \mathfrak{Y}_\psi$  and  $\Lambda(\mathfrak{Y}_\eta) \cap \Lambda(\mathfrak{Y}_\psi) = \emptyset$ . So,  $\Lambda(\mathfrak{Y}_\eta)$  and  $\Lambda(\mathfrak{Y}_\psi)$  are disjoint  $\mathfrak{N}_{\overline{\mathcal{D}}\mathfrak{D}}$ -open sub sets of  $\Lambda(\mathcal{U}_a)$ .

Hence,  $\Lambda(\mathcal{U}_a)$  is  $\mathfrak{N}_{\overline{\mathcal{D}}\mathfrak{D}}\mathfrak{N}\ell$ -space.

**Conclusion**

This study examines  $\mathfrak{N}_{\overline{\mathcal{D}}\mathfrak{D}}\mathfrak{N}\ell$ ,  $\mathfrak{N}_{\overline{\mathcal{D}}\mathfrak{D}}\mathfrak{N}\ell$ -spaces and  $\mathfrak{N}_{\overline{\mathcal{D}}\mathfrak{D}}\mathfrak{N}\ell$ -spaces generated through  $\mathfrak{N}_{\overline{\mathcal{D}}\mathfrak{D}}$ ,  $\mathfrak{N}_{\overline{\mathcal{D}}\mathfrak{D}}$ -open sets in  $\mathfrak{D}_\mathfrak{N}$ -Nano topological spaces. Moreover, beneficial results are investigated by contrasting  $\mathfrak{N}_{\overline{\mathcal{D}}\mathfrak{D}}\mathfrak{N}\ell$  and  $\mathfrak{N}_{\overline{\mathcal{D}}\mathfrak{D}}\mathfrak{N}\ell$  and  $\mathfrak{N}_{\overline{\mathcal{D}}\mathfrak{D}}\mathfrak{N}\ell$  spaces in the context of these specific concepts.

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